1

MATRICES AND DETERMINANTS

Animation 1.1 : Matrix
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1. Matrices and Determinants eLearn.Punjab

Students Learning Outcomes

After studying this unit , the students will be able to:

1. Define
- a matrix with real entries and relate its rectangular layout (formation) with real life,
- rows and columns of a matrix,
- the order of a matrix,
- equality of two matrices.
2. Define and identify row matrix, column matrix, rectangular matrix, square matrix, zero/null matrix, diagonal matrix, scalar matrix, identity matrix, transpose of a matrix, symmetric and skew-symmetric matrices.
3. Know whether the given matrices are suitable for addition/ subtraction.
4. Add and subtract matrices.
5. Multiply a matrix by a real number.
6. Verify commutative and associative laws under addition.
7. Define additive identity of a matrix.
8. Find additive inverse of a matrix.
9. Know whether the given matrices are suitable for multiplication.
10. Multiply two (or three) matrices.
11. Verify associative law under multiplication.
12. Verify distributive laws.
13. Show with the help of an example that commutative law under multiplication does not hold in general (i.e., $AB \neq BA$).
14. Define multiplicative identity of a matrix.
15. Verify the result $(AB)^t = B^tA^t$.
16. Define the determinant of a square matrix.
17. Evaluate determinant of a matrix.
18. Define singular and non-singular matrices.
19. Define adjoint of a matrix.
20. Find multiplicative inverse of a non-singular matrix A and verify that $AA^{-1} = I = A^{-1}A$ where I is the identity matrix.
21. Use adjoint method to calculate inverse of a non-singular matrix.

22. Verify the result $(AB)^{-1} = B^{-1}A^{-1}$
23. Solve a system of two linear equations and related real life problems in two unknowns using
- Matrix inversion method,
- Cramer' s rule.

Introduction

The matrices and determinants are used in the field of Mathematics, Physics, Statistics, Electronics and other branches of science. The matrices have played a very important role in this age of Computer Science.

The idea of matrices was given by Arthur Cayley, an English mathematician of nineteenth century, who first developed, "Theory of Matrices" in 1858.

1.1 Matrix

A rectangular array or a formation of a collection of real numbers, say 0, 1, 2, 3, 4 and 7,such as, $\begin{matrix} 1 & 3 & 4 \\ 7 & 2 & 0 \end{matrix}$ and then enclosed by brackets `[ ]' is said to form a matrix $\begin{bmatrix} 1 & 3 & 4 \\ 7 & 2 & 0 \end{bmatrix}$ Similarly $\begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}$ is another matrix.

We term the real numbers used in the formation of a matrix as entries or elements of the matrix. (Plural of matrix is matrices) The matrices are denoted conventionally by the capital letters A, B, C, M, N etc, of the English alphabets.

1.1.1 Rows and Columns of a Matrix

It is important to understand an entity of a matrix with the following formation

In matrix A, the entries presented in horizontal way are called rows.
In matrix A, there are three rows as shown by R_1, R_2 and R_3 of the matrix A.

$$B = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}$$
$C_1 \quad C_2 \quad C_3$

In matrix B, all the entries presented in vertical way are called columns of the matrix B.
In matrix B, there are three columns as shown by C_1, C_2 and C_3.

It is interesting to note that all rows have same number of elements and all columns have same number of elements but number of elements in rows and columns may not be same.

1.1.2 Order of a Matrix

The number of rows and columns in a matrix specifies its order. If a matrix M has m rows and n columns, then M is said to be of order m-by-n. For example,

$M = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ is of order 2-by-3, since it has two rows and three

columns, whereas the matrix $N = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 2 & 3 & 7 \end{bmatrix}$ is a 3-by-3 matrix and

P = [3 2 5] is a matrix of order 1-by-3.

1.1.3 Equal Matrices

Let A and B be two matrices. Then A is said to be equal to B, and denoted by A = B, if and only if;

(i) the order of A = the order of B

(ii) their corresponding entries are equal.

Examples

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(i) $A = \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2+1 \\ -4 & 4-2 \end{bmatrix}$ are equal matrices.

We see that:

(a) the order of matrix A = the order of matrix B

(b) their corresponding elements are equal. Thus A = B

(ii) $L = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $M = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ are not equal matrices.

We see that order of L = order of M but entries in the second row and second column are not same, so L ≠ M.

(iii) $P = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 0 \end{bmatrix}$ are not equal matrices. We see that order of P ≠ order of Q, so P ≠ Q.

EXERCISE 1.1

1. Find the order of the following matrices.

$A = \begin{bmatrix} 2 & 3 \\ -5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 3 & 5 \end{bmatrix}$, $C = [2 \quad 4]$,

$D = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$, $E = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$, $F = [2]$

$G = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$, $H = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 6 \end{bmatrix}$

2. Which of the following matrices are equal?

$A = [3]$, $B = [3 \quad 5]$, $C = [5-2]$,

$D = [5 \quad 3]$, $E = \begin{bmatrix} 4 & 0 \\ 6 & 2 \end{bmatrix}$, $F = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$,

$G = \begin{bmatrix} 3-1 \\ 3+3 \end{bmatrix}$, $H = \begin{bmatrix} 4 & 0 \\ 6 & 2 \end{bmatrix}$, $I = [3 \quad 3+2]$,

$J = \begin{bmatrix} 2+2 & 2-2 \\ 2+4 & 2+0 \end{bmatrix}$

3. Find the values of a, b, c and d which satisfy the matrix equation

$$\begin{bmatrix} a+c & a+2b \\ c-1 & 4d-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2d \end{bmatrix}$$

1.2 Types of Matrices

(i) Row Matrix

A matrix is called a row matrix, if it has only one row. e.g., the matrix M = $[2 \ \ -1 \ \ 7]$ is a row matrix of order 1-by-3 and M = $[1 \ \ -1]$ is a row matrix of order 1-by-2.

(ii) Column Matrix

A matrix is called a column matrix, if it has only one column.

e.g., M = $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and N = $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ are column matrices of order 2-by-1 and 3-by-1 respectively.

(iii) Rectangular Matrix

A matrix M is called rectangular, if the number of rows of M is not equal to the number of M columns.

e.g.,A = $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$; B = $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$; C = $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and D = $\begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix}$

are all rectangular matrices. The order of A is 3-by-2, the order of B is 2-by-3, the order of C is 1-by-3 and order of D is 3-by-1, which indicates that in each matrix the number of rows ≠ the number of columns.

(iv) Square Matrix

A matrix is called a square matrix, if its number of rows is equal to its number of columns.

e.g., A = $\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$, B= $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ and C = $[3]$

are square matrices of orders, 2-by-2, 3-by-3 and 1-by-1 respectively.

(v) Null or Zero Matrix

A matrix is called a null or zero matrix, if each of its entries is 0.

e.g., $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are null matrices of orders 2-by-2, 1-by-2, 2-by-1, 2-by-3 and 3-by-3 respectively. Note that null matrix is represented by O.

(vi) Transpose of a Matrix

A matrix obtained by interchanging the rows into columns or columns into rows of a matrix is called transpose of that matrix. If A is a matrix, then its transpose is denoted by A^t.

e.g., (i) If A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ -1 & 4 & -2 \end{bmatrix}$, then A^t = $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ 3 & 0 & -2 \end{bmatrix}$

(ii) If B = $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$ then B^t = $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$

(iii) If C = $[0 \ \ 1]$, then C^t = $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

If a matrix A is of order 2-by-3, then order of its transpose A^t is 3-by-2.

(vii) Negative of a Matrix

Let A be a matrix. Then its negative, –A is obtained by changing the signs of all the entries of A, i.e.,

If A = $\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, then –A = $\begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$.

(viii) Symmetric Matrix

A square matrix is symmetric if it is equal to its transpose i.e., matrix A is symmetric, if $A^t = A$.

e.g., (i) If $M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 0 \end{bmatrix}$ is a square matrix, then

$M^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 0 \end{bmatrix} = M$. Thus M is a symmetric matrix.

(ii) If $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$, then $A^t = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 1 \\ 3 & 2 & 3 \end{bmatrix}$, $\neq A$

Hence A is not a symmetric matrix.

(ix) Skew-Symmetric Matrix

A square matrix A is said to be skew-symmetric, if $A^t = -A$.

e.g., if $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix}$

then $A^t = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -3 \\ -(-2) & 0 & -1 \\ -(-3) & -(-1) & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} = -A$

Since $A^t = -A$, therefore A is a skew-symmetric matrix.

(x) Diagonal Matrix

A square matrix A is called a diagonal matrix if atleast any one of the entries of its diagonal is not zero and non-diagonal entries are zero.

e.g., $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ are all diagonal matrices of order 3-by-3.

$M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ are diagonal matrices of order 2-by-2.

(xi) Scalar Matrix

A diagonal matrix is called a scalar matrix, if all the diagonal entries are same and non-zero.

For example $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ where k is a constant $\neq 0,1$.

Also $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $C = [5]$ are scalar matrices of order 3-by-3, 2-by-2 and 1-by-1 respectively.

(xii) Identity Matrix

A diagonal matrix is called identity (unit) matrix, if all diagonal entries are 1. It is denoted by I.

e.g., $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3-by-3 identity matrix, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a 2-by-2 identity matrix, and $C = [1]$ is a 1-by-1 identity matrix.

Note: (i) A scalar and identity matrix are diagonal matrices.
(ii) A diagonal matrix is not a scalar or identity matrix.

EXERCISE 1.2

1. From the following matrices, identify unit matrices, row matrices, column matrices and null matrices.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $B = [2 \quad 3 \quad 4]$, $C = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$,

$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E = [\,0\,]$, $F = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

2. From the following matrices, identify
   (a) Square matrices (b) Rectangular matrices
   (c) Row matrices (d) Column matrices
   (e) Identity matrices (f) Null matrices

(i) $\begin{bmatrix} -8 & 2 & 7 \\ 12 & 0 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

(vi) $\begin{bmatrix} 3 & 10 & -1 \end{bmatrix}$ (vii) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ix) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. From the following matrices, identify diagonal, scalar and unit (identity) matrices.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 5-3 & 0 \\ 0 & 1+1 \end{bmatrix}$$

4. Find negative of matrices A, B, C, D and E when:

$$A = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 6 \\ 3 & 2 \end{bmatrix},$$

$$D = \begin{bmatrix} -3 & 2 \\ -4 & 5 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}$$

5. Find the transpose of each of the following matrices:

$$A = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \quad B = [5 \quad 1 \quad -6], \quad C = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}, \quad E = \begin{bmatrix} 2 & 3 \\ -4 & 5 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

6. Verify that if $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, then

(i) $(A^t)^t = A$ (ii) $(B^t)^t = B$

1.3 Addition and Subtraction of Matrices

1.3.1 Addition of Matrices

Let A and B be any two matrices. The matrices A and B are conformable for addition, if they have the same order.

e.g., $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$ are conformable for addition

Addition of A and B, written A + B is obtained by adding the entries of the matrix A to the corresponding entries of the matrix B.

$$\text{e.g., } A + B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & 6 \end{bmatrix} + \begin{bmatrix} -2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+(-2) & 3+3 & 0+4 \\ 1+1 & 0+2 & 6+3 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 4 \\ 2 & 2 & 9 \end{bmatrix}$$

1.3.2 Subtraction of Matrices

If A and B are two matrices of same order, then subtraction of matrix B from matrix A is obtained by subtracting the entries of matrix B from the corresponding entries of matrix A and it is denoted by A – B.

e.g., $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 2 \\ -1 & 4 & 3 \end{bmatrix}$ are conformable for subtraction.

$$\text{i.e., } A - B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 2 \\ -1 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2-0 & 3-2 & 4-2 \\ 1-(-1) & 5-4 & 0-3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 1 & -3 \end{bmatrix}$$

Some solved examples regarding addition and subtraction are given below.

(a) If $A = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & 3 \\ 2 & 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 & 4 \\ 1 & -1 & 2 \\ 5 & -2 & 7 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & 3 \\ 2 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 4 \\ 1 & -1 & 2 \\ 5 & -2 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 2+3 & 7+4 \\ 0+1 & -1+(-1) & 3+2 \\ 2+5 & 5-2 & 1+7 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 11 \\ 1 & -2 & 5 \\ 7 & 3 & 8 \end{bmatrix}.$$

$$\text{and } A - B = A + (-B) = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & 3 \\ 2 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -3 & -4 \\ -1 & 1 & -2 \\ -5 & 2 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 2-3 & 7-4 \\ 0-1 & -1+1 & 3-2 \\ 2-5 & 5+2 & 1-7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 0 & 1 \\ -3 & 7 & -6 \end{bmatrix}.$$

(b) If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & -2 \\ 3 & 4 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+3 \\ -1+1 & 3-2 \\ 0+3 & 2+4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 3 & 6 \end{bmatrix}.$$

$$\text{and} \quad A - B = \begin{bmatrix} 1-2 & 2-3 \\ -1-1 & 3+2 \\ 0-3 & 2-4 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & 5 \\ -3 & -2 \end{bmatrix}.$$

Note that the order of a matrix is unchanged under the operation of matrix addition and matrix subtraction.

1.3.3 Multiplication of a Matrix by a Real Number

Let A be any matrix and the real number k be a scalar. Then the scalar multiplication of matrix A with k is obtained by multiplying each entry of matrix A with k. It is denoted by kA.

Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & -1 & 0 \\ -1 & 3 & 2 \end{bmatrix}$ be a matrix of order 3-by-3 and $k = -2$ be a real number.

Then,

$$KA = (-2)A = (-2)\begin{bmatrix} 1 & -1 & 4 \\ 2 & -1 & 0 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} (-2)(1) & (-2)(-1) & (-2)(4) \\ (-2)(2) & (-2)(-1) & (-2)(0) \\ (-2)(-1) & (-2)(3) & (-2)(2) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 & -8 \\ -4 & 2 & 0 \\ 2 & -6 & -4 \end{bmatrix}$$

Scalar multiplication of a matrix leaves the order of the matrix unchanged.

1.3.4 Commutative and Associative Laws of Addition of Matrices

(a) Commutative Law under Addition

If A and B are two matrices of the same order, then A + B = B + A is called commulative law under addition.

$$\text{Let } A = \begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & 1 \\ 4 & 2 & -4 \end{bmatrix}$$

$$\text{then } A + B = \begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & 1 \\ 4 & 2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3 & 3-2 & 0+5 \\ 5-1 & 6+4 & 1+1 \\ 2+4 & 1+2 & 3-4 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 5 \\ 4 & 10 & 2 \\ 6 & 3 & -1 \end{bmatrix}$$

Similarly

$$B + A = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & 1 \\ 4 & 2 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 5 \\ 4 & 10 & 2 \\ 6 & 3 & -1 \end{bmatrix}$$

Thus the commutative law of addition of matrices is verified:

$$A + B = B + A$$

(b) Associative Law under Addition

If A, B and C are three matrices of same order, then (A + B) + C = A + (B + C) is called associative law under addition.

$$\text{Let } A = \begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & 1 \\ 4 & 2 & -4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 4 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\text{then } (A + B) + C = \left(\begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & 1 \\ 4 & 2 & -4 \end{bmatrix} \right) + \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 4 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3 & 3-2 & 0+5 \\ 5-1 & 6+4 & 1+1 \\ 2+4 & 1+2 & 3-4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 4 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 & 5 \\ 4 & 10 & 2 \\ 6 & 3 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 4 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 8 \\ 2 & 10 & 6 \\ 7 & 5 & -1 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} + \left(\begin{bmatrix} 3 & -2 & 5 \\ -1 & 4 & 1 \\ 4 & 2 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 4 \\ 1 & 2 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 3+1 & -2+2 & 5+3 \\ -1-2 & 4+0 & 1+4 \\ 4+1 & 2+2 & -4+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 0 \\ 5 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 8 \\ -3 & 4 & 5 \\ 5 & 4 & -4 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 8 \\ 2 & 10 & 6 \\ 7 & 5 & -1 \end{bmatrix}$$

Thus the associative law of addition is verified: (A + B) + C = A + (B + C)

1.3.5 Additive Identity of a Matrix

If A and B are two matrices of same order and A + B = A = B + A, then matrix B is called additive identity of matrix A.
For any matrix A and zero matrix O of same order, O is called additive identity of A as

$$A + O = A = O + A$$

e.g., let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

then $A + O = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = A$

$O + A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = A$

1.3.6 Additive Inverse of a Matrix

If A and B are two matrices of same order such that

A+B=O=B+A,

then A and B are called additive inverses of each other.

Additive inverse of any matrix A is obtained by changing to negative of the symbols (entries) of each non zero entry of A.

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & 0 \end{bmatrix}$

then $B = (-A) = -\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 1 & 2 \\ -3 & -1 & 0 \end{bmatrix}$

is additive inverse of A.
It can be verified as

$$A + B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -2 & -1 \\ 0 & 1 & 2 \\ -3 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1)+(-1) & (2)+(-2) & (1)+(-1) \\ 0+0 & (-1)+(1) & (-2)+(2) \\ (3)+(-3) & (1)+(-1) & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$$B + A = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 1 & 2 \\ -3 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)+(1) & (-2)+(2) & (-1)+(1) \\ 0+0 & (1)+(-1) & (2)+(-2) \\ (-3)+(3) & (-1)+(1) & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Since A + B = O = B + A .
Therefore, A and B are additive inverses of each other.

EXERCISE 1.3

1. Which of the following matrices are conformable for addition?

$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 2+1 \\ 3 \end{bmatrix}$,

$E = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, $F = \begin{bmatrix} 3 & 2 \\ 1+1 & -4 \\ 3+2 & 2+1 \end{bmatrix}$

2. Find additive inverse of the following matrices:

$A = \begin{bmatrix} 2 & 4 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$,

$D = \begin{bmatrix} 1 & 0 \\ -3 & -2 \\ 2 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{2} \end{bmatrix}$

3. If $A = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find,

(i) $A+\begin{bmatrix}1&1\\1&1\end{bmatrix}$ (ii) $B+\begin{bmatrix}-2\\3\end{bmatrix}$ (iii) c=+[-2 1 3]

(iv) $D+\begin{bmatrix}0&1&0\\2&0&1\end{bmatrix}$ (v) 2A (vi) (–1)B

(vii) (–2) C (viii) 3D (ix) 3C

4. Perform the indicated operations and simplify the following:

(i) $\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}0&2\\3&0\end{bmatrix}\right)+\begin{bmatrix}1&1\\1&0\end{bmatrix}$ (ii) $\begin{bmatrix}1&0\\0&1\end{bmatrix}+\left(\begin{bmatrix}0&2\\3&0\end{bmatrix}-\begin{bmatrix}1&1\\1&0\end{bmatrix}\right)$

(iii) [2 3 1] + ([1 0 2] – [2 2 2]) (iv) $=\begin{bmatrix}1&2&3\\-1&-1&-1\\0&1&2\end{bmatrix}+\begin{bmatrix}1&1&1\\2&2&2\\3&3&3\end{bmatrix}$

(v) $\begin{bmatrix}1&2&3\\2&3&1\\3&1&2\end{bmatrix}+\begin{bmatrix}1&0&-2\\-2&-1&0\\0&2&-1\end{bmatrix}$ (vi) $\left(\begin{bmatrix}1&2\\0&1\end{bmatrix}+\begin{bmatrix}2&1\\1&0\end{bmatrix}\right)+\begin{bmatrix}1&1\\1&1\end{bmatrix}$

5. For the matrices $A=\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}$ $B=\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}$ and $C=\begin{bmatrix}-1&0&0\\0&-2&3\\1&1&2\end{bmatrix}$

verify the following rules.

(i) A + C = C + A (ii) A + B = B + A
(iii) B + C = C + B (iv) A + (B + A) = 2A + B
(v) (C – B) + A = C + (A – B) (vi) 2A + B = A + (A + B)
(vii) (C–B) A = (C – A) – B (viii) (A + B) + C = A + (B + C)
(ix) A + (B – C) = (A – C) + B (x) 2A + 2B = 2(A + B)

6. If $A=\begin{bmatrix}1&-2\\3&4\end{bmatrix}$ and $B=\begin{bmatrix}0&7\\-3&8\end{bmatrix}$, find (i) 3A – 2B

(ii) $2A^t-3B^t$.

7. If $2\begin{bmatrix}2&4\\-3&a\end{bmatrix}+3\begin{bmatrix}1&b\\8&-4\end{bmatrix}=\begin{bmatrix}7&10\\18&1\end{bmatrix}$, then find a and b.

8. If $A=\begin{bmatrix}1&2\\0&1\end{bmatrix}$, $B=\begin{bmatrix}1&1\\2&0\end{bmatrix}$, then verify that

(i) $(A+B)^t=A^t+B^t$ (ii) $(A-B)^t=A^t-B^t$
(iii) $A+A^t$ is symmetric (iv) $A-A^t$ is skew symmetric
(v) $B+B^t$ is symmetric (vi) $B-B^t$ is skew symmetric

1.4 Multiplication of Matrices

Two matrices A and B are conformable for multiplication, giving product AB, if the number of columns of A is equal to the number of rows of B.

e.g., let $A=\begin{bmatrix}1&2\\3&0\end{bmatrix}$ and $B=\begin{bmatrix}4\\1\end{bmatrix}$. Here number of columns of A is equal to the number of rows of B. So A and B matrices are conformable for multiplication.

Multiplication of two matrices is explained by the following examples.

(i) If A = [1 2] and $B=\begin{bmatrix}2&0\\3&1\end{bmatrix}$ then $AB=\begin{bmatrix}1&2\end{bmatrix}\begin{bmatrix}2&0\\3&1\end{bmatrix}$

$=[1\times2+2\times3 \quad 1\times0+2\times1]=[2+6 \quad 0+2]=[8 \quad 2]$, is a 1-by-2 matrix.

(ii) If $A=\begin{bmatrix}1&3\\2&-3\end{bmatrix}$ and $B=\begin{bmatrix}-1&0\\3&2\end{bmatrix}$, then

$$AB=\begin{bmatrix}1&3\\2&-3\end{bmatrix}\times\begin{bmatrix}-1&0\\3&2\end{bmatrix}=\begin{bmatrix}1\times(-1)+3\times3 & 1\times0+3\times2\\2(-1)+(-3)(3) & 2\times0+(-3)(2)\end{bmatrix}$$

$$=\begin{bmatrix}-1+9 & 0+6\\-2-9 & 0-6\end{bmatrix}=\begin{bmatrix}8&6\\-11&-6\end{bmatrix}$$, is a 2-by-2 matrix.

1.4.1 Associative Law under Multiplication

If A, B and C are three matrices conformable for multiplication then associative law under multiplication is given as (AB)C = A(BC)

e.g., $A=\begin{bmatrix}2&3\\-1&0\end{bmatrix}$ $B=\begin{bmatrix}0&1\\3&1\end{bmatrix}$ and $C=\begin{bmatrix}2&2\\-1&0\end{bmatrix}$ then

L.H.S. = (AB)C $=\left(\begin{bmatrix}2&3\\-1&0\end{bmatrix}\begin{bmatrix}0&1\\3&1\end{bmatrix}\right)\begin{bmatrix}2&2\\-1&0\end{bmatrix}$

$$= \begin{bmatrix} 2\times0+3\times3 & 2\times1+3\times1 \\ -1\times0+0\times3 & -1\times1+0\times1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+9 & 2+3 \\ 0+0 & -1+0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 5 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 9\times2+5\times(-1) & 9\times2+5\times0 \\ 0\times2+(-1)\times(-1) & 0\times2+(-1)\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 18-5 & 18+0 \\ 0+1 & 0+0 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ 1 & 0 \end{bmatrix}$$

R.H.S = A(BC) = $\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}\right)$

$$= \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0\times2+1\times(-1) & 0\times2+1\times0 \\ 3\times2+1\times(-1) & 3\times2+1\times0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-1)+3\times5 & 2\times0+3\times6 \\ (-1)(-1)+0\times5 & -1\times0+0\times6 \end{bmatrix} = \begin{bmatrix} -2+15 & 0+18 \\ 1+0 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 18 \\ 1 & 0 \end{bmatrix} = (AB)C$$

The associative law under multiplication of matrices is verified.

1.4.2 Distributive Laws of Multiplication over Addition and Subtraction

(a) Let A, B and C be three matrices. Then distributive laws of multiplication over addition are given below:

(i) A(B + C) = AB + AC (Left distributive law)

(ii) (A + B)C = AC + BC (Right distributive law)

Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$, then in (i)

L.H.S = A (B+C)

$$= \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}+\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0+2 & 1+2 \\ 3-1 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2\times2+3\times2 & 2\times3+3\times1 \\ -1\times2+0\times2 & -1\times3+0\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+6 & 6+3 \\ -2+0 & -3+0 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ -2 & -3 \end{bmatrix}$$

R.H.S. = AB + AC

$$= \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}+\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times0+3\times3 & 2\times1+3\times1 \\ -1\times0+0\times3 & -1\times1+0\times1 \end{bmatrix}+\begin{bmatrix} 2\times2+3\times(-1) & 2\times2+3\times0 \\ -1\times2+0\times(-1) & -1\times2+0\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 5 \\ 0 & -1 \end{bmatrix}+\begin{bmatrix} 1 & 4 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 9+1 & 5+4 \\ 0-2 & -1-2 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ -2 & -3 \end{bmatrix} = \text{L.H.S.}$$

Which shows that

A(B + C) = AB + AC; Similarly we can verify (ii).

(b) Similarly the distributive laws of multiplication over subtraction are as follow.

(i) A(B - C) = AB -AC (ii) (A - B)C = AC - BC

Let $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, then in (i)

L.H.S. = A(B − C)

$$= \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}-\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} -1-2 & 1-1 \\ 1-1 & 0-2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (2)(-3)+(3)(0) & 2(0)+3(-2) \\ (0)(-3)+1\times0 & 0\times0+(1)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -6+0 & 0-6 \\ 0+0 & 0-2 \end{bmatrix} = \begin{bmatrix} -6 & -6 \\ 0 & -2 \end{bmatrix}$$

R.H.S. = AB – AC

$$= \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}-\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-1)+3(1) & 2(1)+3(0) \\ 0(-1)+1(1) & 0(1)+1(0) \end{bmatrix} - \begin{bmatrix} 2\times2+3\times1 & 2\times1+3\times2 \\ 0\times2+1\times1 & 0\times1+1\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 7 & 8 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1-7 & 2-8 \\ 1-1 & 0-2 \end{bmatrix} = \begin{bmatrix} -6 & -6 \\ 0 & -2 \end{bmatrix}$$

which shows that

A(B – C) = AB – AC; Similarly (ii) can be verified.

1.4.3 Commutative Law of Multiplication of Matrices

Consider the matrices $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, then

$$AB=\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}=\begin{bmatrix} 0\times1+1\times0 & 0\times0+1(-2) \\ 2\times1+3\times0 & 2\times0+3(-2) \end{bmatrix}=\begin{bmatrix} 0 & -2 \\ 2 & -6 \end{bmatrix}$$

$$\text{and } BA=\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}=\begin{bmatrix} 1\times0+0\times2 & 1\times1+0\times3 \\ 0\times0+(-2)\times2 & 0\times1+3(-2) \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -4 & -6 \end{bmatrix}$$

Which shows that, AB ≠ BA

Commutative law under multiplication in matrices does not hold in general i.e., if A and B are two matrices, then AB ≠ BA.

Commutative law under multiplication holds in particular case.

e.g., if $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}$ then

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times(-3)+0\times0 & 2\times0+0\times4 \\ 0\times(-3)+1\times0 & 0\times0+1\times4 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\text{and } BA = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3\times2+0\times0 & -3\times0+0\times1 \\ 0\times2+4\times0 & 0\times0+4\times1 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 0 & 4 \end{bmatrix}$$

Which shows that AB = BA.

1.4.4 Multiplicative Identity of a Matrix

Let A be a matrix. Another matrix B is called the identity matrix of A under multiplication if

AB = A = BA

If $A = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\times1+2\times0 & 1\times0+2\times1 \\ 0\times1+(-3)\times0 & 0\times0+(-3)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1\times1+0\times0 & 1\times2+0\times(-3) \\ 0\times1+1\times0 & 0\times2+1\times(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$$

Which shows that AB = A = BA.

1.4.5 Verification of $(AB)^t = B^t A^t$

If A, B are two matrices and A^t, B^t are their respective transpose, then $(AB)^t = B^tA^t$.

e.g., $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$

L.H.S. = $(AB)^t$

$$= \left(\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}\right)^t = \begin{bmatrix} 2\times1+1\times(-2) & 2\times3+1\times0 \\ 0\times1+(-1)\times(-2) & 0\times3+(-1)\times0 \end{bmatrix}^t$$

$$= \begin{bmatrix} 2-2 & 6+0 \\ 0+2 & 0+0 \end{bmatrix}^t = \begin{bmatrix} 0 & 6 \\ 2 & 0 \end{bmatrix}^t = \begin{bmatrix} 0 & 2 \\ 6 & 0 \end{bmatrix}$$

R.H.S. = $B^t A^t$,

$$(A)^t = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}^t = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \text{ and } (B)^t = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}^t = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$$

$$B^tA^t = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1\times2+(-2)\times1 & 1\times0+(-2)(-1) \\ 3\times2+0\times1 & 3\times0+0\times(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & 0+2 \\ 6+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 6 & 0 \end{bmatrix} = \text{L.H.S.}$$

Thus $(AB)^t = B^tA^t$

EXERCISE 1.4

1. Which of the following product of matrices is conformable for multiplication?

(i) $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -2 & 3 \end{bmatrix}$

2. If $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$, find (i) AB (ii) BA (if possible)

3. Find the following products.

(i) $\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \end{bmatrix}$ (iii) $\begin{bmatrix} -3 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 6 & -0 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 2 \\ -3 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 0 & -4 \end{bmatrix}$

4. Multiply the following matrices.

(a) $\begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 8 & 5 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} 2 & -\frac{5}{2} \\ -4 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

5. Let $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$. verify whether

(i) AB = BA. (ii) A(BC) = (AB)C

(iii) A(B + C) = AB + AC (iv) A(B – C) = AB – AC

6. For the matrices

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 6 \\ 3 & -9 \end{bmatrix}$$

Verify that (i) $(AB)^t = B^t A^t$ (ii) $(BC)^t = C^t B^t$.

1.5 Multiplicative Inverse of a Matrix

1.5.1 Determinant of a 2-by-2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2-by-2 square matrix. The determinant of A, denoted by det A or $|A|$ is defined as

$$|A| = \det A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \lambda \in R$$

e.g., Let $B = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$.

Then $|B| = \det B = \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} = 1 \times 3 - (-2)(1) = 3 + 2 = 5$

If $M = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$, then $\det M = \begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} = 2 \times 3 - 1 \times 6 = 0$

1.5.2 Singular and Non-Singular Matrix

A square matrix A is called singular, if the determinant of A is equal to zero. i.e., $|A| = 0$.

For example, $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ is a singular matrix,

since det A = 1 × 0 – 0 × 2 = 0

A square matrix A is called non-singular, if the determinant of A is not

equal to zero. i.e., $|A| \neq 0$. For example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is non-singular, since det $A = 1 \times 2 - 0 \times 1 = 2 \neq 0$. Note that, each square matrix with real entries is either singular or non-singular.

1.5.3 Adjoint of a Matrix

Adjoint of a square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is obtained by interchanging the diagonal entries and changing the signs of other entries. Adjoint of matrix A is denoted as Adj A.

$$\text{i.e.,} \quad \text{Adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{e.g., if } A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \text{ then Adj } A = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\text{If } B = \begin{bmatrix} 2 & -1 \\ 3 & -4 \end{bmatrix}, \text{ then Adj } B = \begin{bmatrix} -4 & 1 \\ -3 & 2 \end{bmatrix}$$

1.5.4 Multiplicative Inverse of a Non-singular Matrix

Let A and B be two non-singular square matrices of same order. Then A and B are said to be multiplicative inverse of each other if

$$AB = BA = I.$$

The inverse of A is denoted by A^{-1}, thus

$$AA^{-1} = A^{-1}A = I.$$

Inverse of Identity matrix is Identity matrix.

Inverse of a matrix is possible only if matrix is non-singular.

1.5.5 Inverse of a Matrix using Adjoint

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square matrix. To find the inverse of M, i.e., M^{-1}, first we find the determinant as inverse is possible only of a non-singular matrix.

$$|M| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$$

and $\text{Adj } M = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, then $M^{-1} = \dfrac{\text{Adj } M}{|M|}$

e.g., Let $A = \begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}$, Then

$$|A| = \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} = -6 - (-1) = -6 + 1 = -5 \neq 0$$

$$\text{Thus } A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{\begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}}{-5} = \frac{-1}{5}\begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-1}{5} & \frac{-2}{5} \end{bmatrix}$$

$$\text{and } AA^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-1}{5} & \frac{-2}{5} \end{bmatrix} = \begin{bmatrix} \frac{6}{5} - \frac{1}{5} & \frac{2}{5} - \frac{2}{5} \\ -\frac{3}{5} + \frac{3}{5} & -\frac{1}{5} + \frac{6}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = AA^{-1}$$

1.5.6 Verification of $(AB)^{-1} = B^{-1}A^{-1}$

Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}$

Then det $A = 3 \times 0 - (-1) \times 1 = 1 \neq 0$

and det $B = 0 \times 2 - 3(-1) = 3 \neq 0$

Therefore, A and B are invertible i.e., their inverses exist.

Then, to verify the law of inverse of the product, take

$$AB = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 \times 0 + 1 \times 3 & 3 \times (-1) + 1 \times 2 \\ -1 \times 0 + 0 \times 3 & -1 \times (-1) + 0 \times 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \det(AB) = = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} = 3 \neq 0$$

$$\text{and L.H.S.} = (AB)^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & 1 \end{bmatrix}$$

$$\text{R.H.S.} = B^{-1}A^{-1}, \text{ where } B^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}, \quad A^{-1} = \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}2 & 1\\ -3 & 0\end{bmatrix}\cdot\frac{1}{1}\begin{bmatrix}0 & -1\\ 1 & 3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2\times0+1\times1 & 2\times(-1)+1\times3\\ -3\times0+0\times1 & -3\times(-1)+0\times3\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}0+1 & -2+3\\ 0 & 3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1 & 1\\ 0 & 3\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & \frac{1}{3}\\ 0 & 1\end{bmatrix}$$

$= (AB)^{-1}$ Thus the law $(AB)^{-1} = B^{-1}A^{-1}$ is verified.

EXERCISE 1.5

1. Find the determinant of the following matrices.

(i) $A = \begin{bmatrix}-1 & 1\\ 2 & 0\end{bmatrix}$ (ii) $B = \begin{bmatrix}1 & 3\\ 2 & -2\end{bmatrix}$

(iii) $C = \begin{bmatrix}3 & 2\\ 3 & 2\end{bmatrix}$ (iv) $D = \begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}$

2. Find which of the following matrices are singular or non-singular?

(i) $A = \begin{bmatrix}3 & 6\\ 2 & 4\end{bmatrix}$ (ii) $B = \begin{bmatrix}4 & 1\\ 3 & 2\end{bmatrix}$

(iii) $C = \begin{bmatrix}7 & -9\\ 3 & 5\end{bmatrix}$ (iv) $D = \begin{bmatrix}5 & -10\\ -2 & 4\end{bmatrix}$

3. Find the multiplicative inverse (if it exists) of each.

(i) $A = \begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}$ (ii) $B = \begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}$

(iii) $C = \begin{bmatrix}-2 & 6\\ 3 & -9\end{bmatrix}$ (iv) $D = \begin{bmatrix}\frac{1}{2} & \frac{3}{4}\\ 1 & 2\end{bmatrix}$

4. If $A = \begin{bmatrix}1 & 2\\ 4 & 6\end{bmatrix}$ and $B = \begin{bmatrix}3 & -1\\ 2 & -2\end{bmatrix}$, then

(i) $A(\text{Adj } A) = (\text{Adj } A)A = (\det A)I$ (ii) $BB^{-1} = I = B^{-1}B$

5. Determine whether the given matrices are multiplicative inverses of each other.

(i) $\begin{bmatrix}3 & 5\\ 4 & 7\end{bmatrix}$ and $\begin{bmatrix}7 & -5\\ -4 & 3\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}$ and $\begin{bmatrix}-3 & 2\\ 2 & -1\end{bmatrix}$

6. If $A = \begin{bmatrix}4 & 0\\ -1 & 2\end{bmatrix}$, $B = \begin{bmatrix}-4 & -2\\ 1 & -1\end{bmatrix}$, $D = \begin{bmatrix}3 & 1\\ -2 & 2\end{bmatrix}$, then verify that

(i) $(AB)^{-1} = B^{-1}A^{-1}$ (ii) $(DA)^{-1} = A^{-1}D^{-1}$

1.6 Solution of Simultaneous Linear Equations

System of two linear equations in two variables in general form is given as

$$ax + by = m$$
$$cx + dy = n$$

where *a, b, c, d, m and n* are real numbers.

This system is also called simultaneous linear equations.

We discuss here the following methods of solution.

(i) **Matrix** inversion method

(ii) **Cramer's** rule

(i) Matrix Inversion Method

Consider the system of linear equations

$$ax + by = m$$
$$cx + dy = n$$

Then $\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}m\\ n\end{bmatrix}$

or $AX = B$

where $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$, $X = \begin{bmatrix}x\\ y\end{bmatrix}$ and $B = \begin{bmatrix}m\\ n\end{bmatrix}$

or $X = A^{-1}B$ $\quad |A| = ad - bc$

or $X = \dfrac{\text{Adj } A}{|A|}\times B \quad \because A^{-1} = \dfrac{\text{Adj } A}{|A|}$ and $|A| \neq 0$

$$\text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix}}{ad-bc}$$

$$= \begin{bmatrix} \dfrac{dm-bn}{ad-bc} \\ \dfrac{-cm+an}{ad-bc} \end{bmatrix}$$

$$\Rightarrow \quad x = \frac{dm-bn}{ad-bc} \quad \text{and} \quad y = \frac{an-cm}{ad-bc}$$

(ii) Cramer's Rule

Consider the following system of linear equations.

$ax + by = m$

$cx + dy = n$

We know that

$$AX = B, \text{ where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} m \\ n \end{bmatrix}$$

$$\text{or} \quad X = A^{-1}B \quad \text{or} \quad X = \frac{\text{Adj } A}{|A|} \times B$$

$$\text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix}}{|A|} = \frac{\begin{bmatrix} dm-bn \\ -cm+an \end{bmatrix}}{|A|}$$

$$= \begin{bmatrix} \dfrac{dm-bn}{|A|} \\ \dfrac{-cm+an}{|A|} \end{bmatrix}$$

$$\text{or} \quad x = \frac{dm-bn}{|A|} = \frac{|A_x|}{|A|}$$

$$\text{and} \quad y = \frac{an-cm}{|A|} = \frac{|A_y|}{|A|}$$

$$\text{where} \quad |A_x| = \begin{vmatrix} m & b \\ n & d \end{vmatrix} \quad \text{and} \quad |A_y| = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$$

Example 1

Solve the following system by using matrix inversion method.

$4x - 2y = 8$

$3x + y = -4$

Solution

Step 1 $\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$

Step 2 The coefficient matrix $M = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$ is non-singular,

since det M = $4 \times 1 - 3(-2) = 4 + 6 = 10 \neq 0$. So M^{-1} is possible.

Step 3 $\begin{bmatrix} x \\ y \end{bmatrix} = M^{-1}\begin{bmatrix} 8 \\ -4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 8 \\ -4 \end{bmatrix}$

$$= \frac{1}{10}\begin{bmatrix} 8-8 \\ -24-16 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 0 \\ -40 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

$$\Rightarrow \quad x = 0 \text{ and } y = -4$$

Example 2

Solve the following system of linear equations by using Cramer's rule.

$3x - 2y = 1$

$-2x + 3y = 2$

Solution

$3x - 2y = 1$

$-2x + 3y = 2$

We have

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}, A_x = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}, A_y = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = 9 - 4 = 5 \neq 0 \quad \text{(A is non-singular}$$

$$x = \frac{|A_x|}{|A|} = \frac{\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}}{5} = \frac{3+4}{5} = \frac{7}{5}$$

$$y = \frac{|A_y|}{|A|} = \frac{\begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix}}{5} = \frac{6+2}{5} = \frac{8}{5}$$

Example 3

The length of a rectangle is 6 cm less than three times its width. The perimeter of the rectangle is 140 cm. Find the dimensions of the rectangle. (by using matrix inversion method)

Solution

If width of the rectangle is x cm, then length of the rectangle is

$y = 3x - 6$,

from the condition of the question.

The perimeter = $2x + 2y = 140$ *(According to given condition)*

$\Rightarrow \quad x + y = 70$ ……(i)

and $3x - y = 6$ ……(ii)

In the matrix form

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 70 \\ 6 \end{bmatrix}$$

$$\det\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = 1\times(-1) - 3\times 1 = -1-3 = -4 \neq 0$$

We know that

$$X = A^{-1}B \quad \text{and} \quad A^{-1} = \frac{\text{Adj}A}{|A|}$$

Hence $$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-4}\begin{vmatrix} -1 & -1 \\ -3 & 1 \end{vmatrix}\begin{bmatrix} 70 \\ 6 \end{bmatrix}$$

$$= \frac{-1}{4}\begin{vmatrix} -70-6 \\ -210+6 \end{vmatrix} = \frac{-1}{4}\begin{bmatrix} -76 \\ -204 \end{bmatrix} = \begin{bmatrix} \frac{76}{4} \\ \frac{204}{4} \end{bmatrix} = \begin{bmatrix} 19 \\ 51 \end{bmatrix}$$

Thus, by the equality of matrices, width of the rectangle x = 19 cm and the length y = 51 cm.

Verification of the solution to be correct, i.e.,

p = 2 × 19 + 2 × 51 = 38 + 102= 140 cm

Also y = 3(19) – 6 = 57 – 6 = 51 cm

EXERCISE 1.6

1 Use matrices, if possible, to solve the following systems of linear equations by:

(i) the matrix inversion method (ii) the Cramer's rule.

(i) $2x - 2y = 4$, $3x + 2y = 6$ (ii) $2x + y = 3$, $6x + 5y = 1$

(iii) $4x + 2y = 8$, $3x - y = -1$ (iv) $3x - 2y = -6$, $5x - 2y = -10$

(v) $3x - 2y = 4$, $-6x + 4y = 7$ (vi) $4x + y = 9$, $-3x - y = -5$

(vii) $2x - 2y = 4$, $-5x - 2y = -10$ (viii) $3x - 4y = 4$, $x + 2y = 8$

Solve the following word problems by using

(i) matrix inversion method (ii) Crammer's rule.

2 The length of a rectangle is 4 times its width. The perimeter of the rectangle is 150 cm. Find the dimensions of the rectangle.

3 Two sides of a rectangle differ by 3.5cm. Find the dimensions of the rectangle if its perimeter is 67cm.

4 The third angle of an isosceles triangle is 16° less than the sum of the two equal angles. Find three angles of the triangle.

5 One acute angle of a right triangle is 12° more than twice the other acute angle. Find the acute angles of the right triangle.

6 Two cars that are 600 km apart are moving towards each other. Their speeds differ by 6 km per hour and the cars are 123 km apart after $4\frac{1}{2}$ hours. Find the speed of each car.

REVIEW EXERCISE 1

2. Complete the following:
   (i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called ..... matrix.
   (ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called ..... matrix.
   (iii) Additive inverse of $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$ is.........
   (iv) In matrix multiplication,in general, AB ...... BA.
   (v) Matrix A + B may be found if order of A and B is ......
   (vi) A matrix is called ..... matrix if number of rows and columns are equal.

3. If $\begin{bmatrix} a+3 & 4 \\ 6 & b-1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$, then find a and b.

4. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -4 \\ -2 & -1 \end{bmatrix}$, then find the following.

   (i) $2A + 3B$ (ii) $-3A + 2B$
   (iii) $-3(A + 2B)$ (iv) $\frac{2}{3}(2A - 3B)$

5. Find the value of X, if $\begin{bmatrix} 2 & 1 \\ 3 & -3 \end{bmatrix} + X = \begin{bmatrix} 4 & -2 \\ -1 & -2 \end{bmatrix}$.

6. If $A = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 \\ 5 & -2 \end{bmatrix}$, then prove that

   (i) $AB \neq BA$ (ii) $A(BC) = (AB)C$

7. If $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -3 & -5 \end{bmatrix}$, then verify that

   (i) $(AB)^t = B^tA^t$ (ii) $(AB)^{-1} = B^{-1}A^{-1}$

SUMMARY

- A rectangular array of real numbers enclosed with brackets is said to form a matrix.
- A matrix A is called rectangular, if the number of rows and number of columns of A are not equal.
- A matrix A is called a square matrix, if the number of rows of A is equal to the number of columns.
- A matrix A is called a row matrix, if A has only one row.
- A matrix A is called a column matrix, if A has only one column.
- A matrix A is called a null or zero matrix, if each of its entry is 0.
- Let A be a matrix. The matrix A^t is a new matrix which is called transpose of matrix A and is obtained by interchanging rows of A into its respective columns (or columns into respective rows).
- A square matrix A is called symmetric, if $A^t = A$.
- Let A be a matrix. Then its negative, –A, is obtained by changing the signs of all the entries of A.
- A square matrix M is said to be skew symmetric, if $M^t = -M$,
- A square matrix M is called a diagonal matrix, if atleast any one of entry of its diagonal is not zero and remaining entries are zero.
- A diagonal matrix is called identity matrix, if all diagonal entries are

  1. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is called a 3-by-3 identity matrix.

- Any two matrices A and B are called equal, if
  (i) order of A= order of B (ii) corresponding entries are same
- Any two matrices M and N are said to be conformable for addition, if order of M = order of N.
- Let A be a matrix of order 2-by-3. Then a matrix B of same order is said to be an additive identity of matrix A, if

  $B + A = A = A + B$

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- Let A be a matrix. A matrix B is defined as an additive inverse of A, if B + A = O = A + B
- Let A be a matrix. Another matrix B is called the identity matrix of A under multiplication, if

$$B \times A = A = A \times B.$$

- Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2-by-2 matrix. A real number λ is called determinant of M, denoted by det M such that

$$\det M = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \lambda$$

- A square matrix M is called singular, if the determinant of M is equal to zero.
- A square matrix M is called non-singular, if the determinant of M is not equal to zero.
- For a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, adjoint of M is defined by

$$\text{Adj } M = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- Let M be a square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$M^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ where } \det M = ad - bc \neq 0.$$

- The following laws of addition hold
  - M + N = N + M (Commutative)
  - (M + N) + T = M + (N + T) (Associative)
- The matrices M and N are conformable for multiplication to obtain MN if the number of columns of M = number of rows of N, where
  - (i) (MN) $\neq$ (NM), in general
  - (ii) (MN)T = M(NT) (Associative law)
  - (iii) M(N + T) = MN + MT } (Distributive laws)
  - (iv) (N + T)M = NM + TM } (Distributive laws)
- Law of transpose of product $(AB)^t = B^t A^t$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $AA^{-1} = I = A^{-1}A$

- The solution of a linear system of equations,

$$ax + by = m$$
$$cx + dy = n$$

by expressing in the matrix form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$

is given by $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} m \\ n \end{bmatrix}$

if the coefficient matrix is non-singular.

- By using the Cramer's rule the determinental form of solution of equations

$$ax + by = m$$
$$cx + dy = n$$

is

$$x = \frac{\begin{vmatrix} m & b \\ n & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a & m \\ c & n \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad \text{where } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

CHAPTER

2

REAL AND COMPLEX NUMBERS

Animation 2.1:Real And Complex numbers
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Students Learning Outcomes

- After studying this unit , the students will be able to:
- Recall the set of real numbers as a union of sets of rational and irrational numbers.
- Depict real numbers on the number line.
- Demonstrate a number with terminating and non-terminating recurring decimals on the number line.
- Give decimal representation of rational and irrational numbers.
- Know the properties of real numbers.
- Explain the concept of radicals and radicands.
- Differentiate between radical form and exponential form of an expression.
- Transform an expression given in radical form to an exponential form and vice versa.
- Recall base, exponent and value.
- Apply the laws of exponents to simplify expressions with real exponents.
- Define complex number z represented by an expression of the form $z = a + ib$, where a and b are real numbers and $i = \sqrt{-1}$
- Recognize *a* as real part and *b* as imaginary part of *z* = *a* + *ib*.
- Define conjugate of a complex number.
- Know the condition for equality of complex numbers.
- Carry out basic operations (i.e., addition, subtraction, multiplication and division) on complex numbers.

Introduction

The numbers are the foundation of mathematics and we use different kinds of numbers in our daily life. So it is necessary to be familiar with various kinds of numbers In this unit we shall discuss real numbers and complex numbers including their properties. There is a one-one correspondence between real numbers and the points on the real line. The basic operations of addition, subtraction, multiplication and division on complex numbers will also be discussed in this unit.

2.1 Real Numbers

We recall the following sets before giving the concept of real numbers.

Natural Numbers

The numbers 1, 2, 3, ... which we use for counting certain objects are called natural numbers or positive integers. The set of natural numbers is denoted by N.

i.e., N = {1,2,3, ...}

Whole Numbers

If we include 0 in the set of natural numbers, the resulting set is the set of whole numbers, denoted by W,

i.e., W = {o,1,2,3, ...}

Integers

The set of integers consist of positive integers, 0 and negative integers and is denoted by Z i.e., Z = { ..., –3, –2, –1, 0, 1, 2, 3, ... }

2.1.1 Set of Real Numbers

First we recall about the set of rational and irrational numbers.

Rational Numbers

All numbers of the form p/q where p, q are integers and q is not zero are called rational numbers. The set of rational numbers is denoted by Q,

$$i.e., \quad Q = \left\{ \frac{p}{q} \mid p, q \in Z \wedge q \neq 0 \right\}$$

Irrational Numbers

The numbers which cannot be expressed as quotient of integers are called irrational numbers.

The set of irrational numbers is denoted by Q′,

$$Q' = \left\{ x \mid x \neq \frac{p}{q}, p, q \in Z \wedge q \neq 0 \right\}$$

For example, the numbers $\sqrt{2}, \sqrt{3}, \sqrt{5}$, π and e are all irrational numbers. The union of the set of rational numbers and irrational numbers is known as the set of real numbers. It is denoted by R,

i.e., $R = Q \cup Q'$

Here Q and Q′ are both subset of R and $Q \cap Q' = \phi$

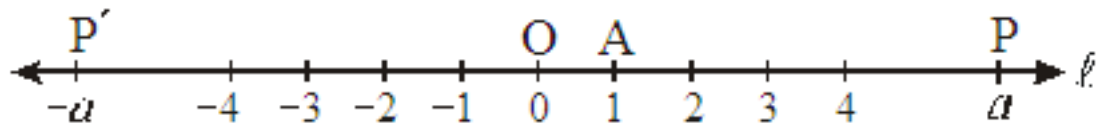
Note:

(i) $N \subset W \subset Z \subset Q$
(ii) Q and Q′ are disjoint sets.
(iii) for each prime number p, $\sqrt{p}$ is an irrational number.
(iv) square roots of all positive non- square integers are irrational.

i.e., $N \subset W \subset Z \subset Q \subset R$

2.1.2 Depiction of Real Numbers on Number Line

The real numbers are represented geometrically by points on a number line l such that each real number '*a*' corresponds to one and only one point on number line l and to each point P on number line l there corresponds precisely one real number. This type of association or relationship is called a one-to-one correspondence. We establish such correspondence as below.

We first choose an arbitrary point O (the origin) on a horizontal line l and associate with it the real number 0. By convention, numbers to the right of the origin are positive and numbers to the left of the origin are negative. Assign the number 1 to the point A so that the line segment OA represents one unit of length.

The number '*a*' associated with a point P on ℓ is called the coordinate of P, and ℓ is called the coordinate line or the real number line. For any real number a, the point P′($-a$) corresponding to $-a$ lies at the same distance from O as the point P (a) corresponding to a but in the opposite direction.

2.1.3 Demonstration of a Number with Terminating and Non-Terminating decimals on the Number Line

First we give the following concepts of rational and irrational numbers.

(a) Rational Numbers

The decimal representations of rational numbers are of two types, terminating and recurring.

(i) Terminating Decimal Fractions

The decimal fraction in which there are finite number of digits in its decimal part is called a terminating decimal fraction. For example

$\frac{2}{5} = 0.4$ and $\frac{3}{8} = 0.375$

(ii) Recurring and Non-terminating Decimal Fractions

The decimal fraction (non-terminating) in which some digits are repeated again and again in the same order in its decimal part is called a recurring decimal fraction. For example

$\frac{2}{9} = 0.2222$ and $\frac{4}{11} = 0.363636...$

(b) Irrational Numbers

It may be noted that the decimal representations for irrational numbers are neither terminating nor repeating in blocks. The decimal form of an irrational number would continue forever and never begin to repeat the same block of digits.

e.g., $\sqrt{2} = 1.414213562...$, $\pi = 3.141592654...$, e= 2.718281829..., etc.

Obviously these decimal representations are neither terminating nor recurring.

We consider the following example.

Example

Express the following decimals in the form $\frac{p}{q}$ where p, q ∈ Z and q ≠ 0

(a) $0.\bar{3} = 0.333...$ (b) $0.\overline{23} = 0.232323...$

Solution

(a) Let $x = 0.\bar{3}$ which can be rewritten as

$x = 0.3333...$ ...... (i)

Note that we have only one digit 3 repeating indefinitely.

So, we multiply both sides of (i) by 10, and obtain

$10x = (0.3333...) \times 10$

or $10x = 3.3333...$ ...... (ii)

Subtracting (i) from (ii), we have

$10x - x = (3.3333...) - (0.3333...)$

or $9x = 3 \Rightarrow x = \frac{1}{3}$

Hence $0.\bar{3} = \frac{1}{3}$

(b) Let $x = 0.\overline{23} = 0.23\ 23\ 23...$

Since two digit block 23 is repeating itself indefinitely, so we multiply both sides by 100 .

Then $100x = 23.\overline{23}$

$100x = 23 + 0.\overline{23} = 23 + x$

$\Rightarrow 100x - x = 23$

$\Rightarrow 99x = 23$

$\Rightarrow x = \frac{23}{99}$

Thus $0.\overline{23} = \frac{23}{99}$ is a rational number.

2.1.4 Representation of Rational and Irrational Numbers on Number Line

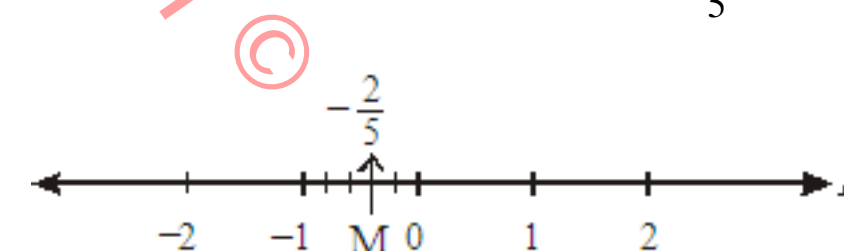
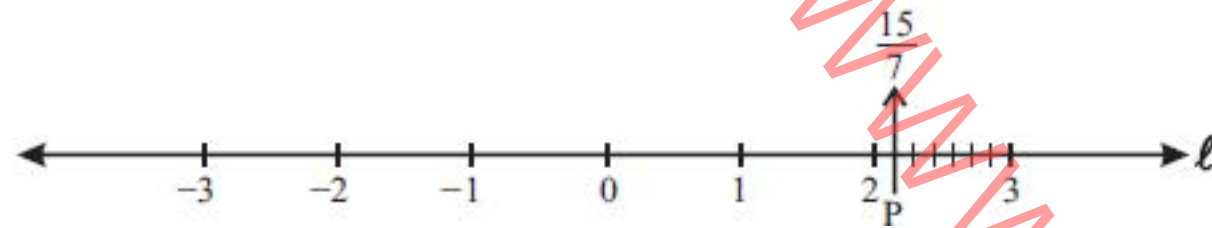
In order to locate a number with terminating and non-terminating recurring decimal on the number line, the points associated with the rational numbers $\frac{m}{n}$ and $-\frac{m}{n}$ where *m, n* are positive integers, we subdivide each unit length into *n* equal parts. Then the *m*th point of division to the right of the origin represents $\frac{m}{n}$ and that to the left of the origin at the same distance represents $-\frac{m}{n}$

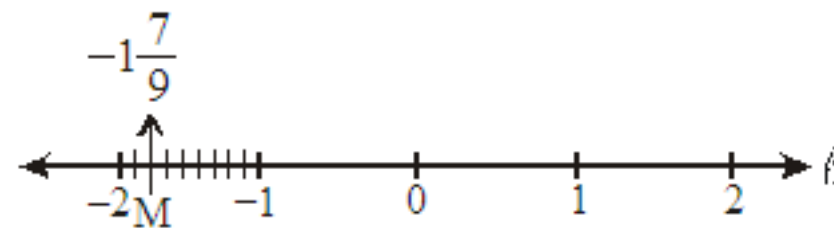
Example

Represent the following numbers on the number line.

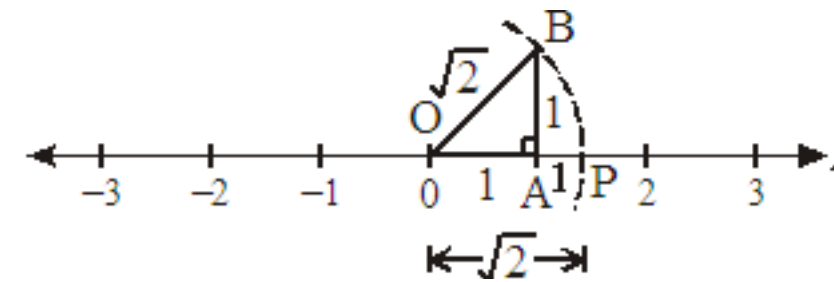
(i) $-\frac{2}{5}$ (ii) $\frac{15}{5}$ (iii) $-1\frac{7}{9}$

Solution

(i) For representing the rational number $-\frac{2}{5}$ on the number line ℓ, divide the unit length between 0 and –1 into five equal parts and take the end of the second part from 0 to its left side. The point M in the following figure represents the rational number $-\frac{2}{5}$

(*ii*) $\frac{15}{7}=2+\frac{1}{7}$: it lies between 2 and 3.

Divide the distance between 2 and 3 into seven equal parts. The point P represents the number $\frac{15}{7}=2\frac{1}{7}$.

(iii) For representing the rational number, $-1\frac{7}{9}$. divide the unit length between –1 and –2 into nine equal parts. Take the end of the 7th part from –1. The point M in the following figure represents the rational number, $-1\frac{7}{9}$.

Irrational numbers such as $\sqrt{2}$, $\sqrt{5}$ etc. can be located on the line ℓ by geometric construction. For example, the point corresponding to $\sqrt{2}$ may be constructed by forming a right ΔOAB with sides (containing the right angle) each of length 1 as shown in the figure. By Pythagoras Theorem,

$$OB=\sqrt{(1)^2+(1)^2}=\sqrt{2}$$

By drawing an arc with centre at O and radius OB = $\sqrt{2}$, we get the point P representing $\sqrt{2}$ on the number line.

EXERCISE 2.1

1. Identify which of the following are rational and irrational numbers.

   (i) $\sqrt{3}$ (ii) $\frac{1}{6}$ (iii) π (iv) $\frac{15}{2}$ (v) 7.25 (vi) $\sqrt{29}$

2. Convert the following fractions into decimal fractions.

   (i) $\frac{17}{25}$ (ii) $\frac{19}{4}$ (iii) $\frac{57}{8}$ (iv) $\frac{205}{18}$ (v) $\frac{5}{8}$ (vi) $\frac{25}{38}$

3. Which of the following statements are true and which are false?

   (i) $\frac{2}{3}$ is an irrational number. (ii) π is an irrational number.

   (iii) $\frac{1}{9}$ is a terminating fraction. (iv) $\frac{3}{4}$ is a terminating fraction.

   (v) $\frac{4}{5}$ is a recurring fraction.

4. Represent the following numbers on the number line.

   (i) $\frac{2}{3}$ (ii) $-\frac{4}{5}$ (iii) $1\frac{3}{4}$ (iv) $-2\frac{5}{8}$ (v) $2\frac{3}{4}$ (vi) $\sqrt{5}$

5. Give a rational number between $\frac{3}{4}$ and $\frac{5}{9}$.
6. Express the following recurring decimals as the rational number $\frac{p}{q}$ where *p*, *q* are integers and $q \neq 0$ (i) $0.\overline{5}$ (ii) $0.\overline{13}$ (iii) $0.\overline{67}$

2.2 Properties of Real Numbers

If a, b are real numbers, their sum is written as $a + b$ and their product as ab or $a \times b$ or $a \cdot b$ or $(a)(b)$.

(a) Properties of Real numbers with respect to Addition and Multiplication Properties of real numbers under addition are as follows:

(i) Closure Property

$$a + b \in R, \quad \forall \; a, b \in R$$

e. g., if -3 and $5 \in R$,
then $-3 + 5 = 2 \in R$

(ii) Commutative Property

$$a + b = b + a, \quad \forall \; a, b \in R$$

e.g., if $2, 3 \in R$,
then $2 + 3 = 3 + 2$
or $5 = 5$

(iii) Associative Property

$$(a + b) + c = a + (b + c), \quad \forall \; a, b, c \in R$$

e.g., if $5, 7, 3 \in R$,
then $(5 + 7) + 3 = 5 + (7 + 3)$
or $12 + 3 = 5 + 10$
or $15 = 15$

(iv) Additive Identity

There exists a unique real number 0, called additive identity, such that

$$a + 0 = a = 0 + a, \quad \forall \; a \in R$$

(v) Additive Inverse

For every $a \in R$, there exists a unique real number $-a$, called the additive inverse of a, such that

$$a + (-a) = 0 = (-a) + a$$

e.g., additive inverse of 3 is -3 since $3 + (-3) = 0 = (-3) + (3)$

Properties of real numbers under multiplication are as follows:

(i) Closure Property

$$ab \in R, \quad \forall \; a, b \in R$$

e.g., if $-3, 5 \in R$,
then $(-3)(5) \in R$
or $-15 \in R$

(ii) Commutative Property

$$ab = ba, \quad \forall \; a, b \in R$$

e.g., if $\frac{1}{3}, \frac{3}{2} \in R$
then $\left(\frac{1}{3}\right)\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)\left(\frac{1}{3}\right)$
or $\frac{1}{2} = \frac{1}{2}$

(i) Associative Property

$$(ab)c = a(bc), \quad \forall \; a, b, c \in R$$

e.g., if $2, 3, 5 \in R$,
then $(2 \times 3) \times 5 = 2 \times (3 \times 5)$
or $6 \times 5 = 2 \times 15$
or $30 = 30$

(ii) Multiplicative Identity

There exists a unique real number 1, called the multiplicative identity, such that

$a \cdot 1 = a = 1 \cdot a, \quad \forall \quad a \in R$

(iii) Multiplicative Inverse

For every non-zero real number, there exists a unique real number a^{-1} or $\frac{1}{a}$, called multiplicative inverse of a, such that

$$aa^{-1} = 1 = a^{-1}a$$

or $\quad a \times \frac{1}{a} = 1 = \frac{1}{a} \times a$

e.g., if $5 \in R$, then $\frac{1}{5} \in R$

such that

$$5 \times \frac{1}{5} = 1 = \frac{1}{5} \times 5$$

So, 5 and $\frac{1}{5}$ are multiplicative inverse of each other.

(vi) Multiplication is Distributive over Addition and Subtraction

For all $a, b, c \in R$

$a(b + c) = ab + ac$ (Left distributive law)

$(a + b)c = ac + bc$ (Right distributive law)

e.g., if $2, 3, 5 \in R$, then

$2(3 + 5) = 2 \times 3 + 2 \times 5$

or $\quad 2 \times 8 = 6 + 10$

or $\quad 16 = 16$

And for all $a, b, c \in R$

$a(b - c) = ab - ac$ (Left distributive law)

$(a - b)c = ac - bc$ (Right distributive law)

e.g., if $2, 5, 3 \in R$, then

$2(5 - 3) = 2 \times 5 - 2 \times 3$

or $\quad 2 \times 2 = 10 - 6$

or $\quad 4 = 4$

Note:

(i) The symbol $\forall$ means "for all",

(ii) a is the multiplicative inverse of a^{-1}, i.e., $a = (a^{-1})^{-1}$

(b) Properties of Equality of Real Numbers

Properties of equality of real numbers are as follows:

(i) Reflexive Property

$a = a, \quad \forall \quad a \in R$

(ii) Symmetric Property

If $a = b$, then $b = a$, $\quad \forall \quad a, b \in R$

(iii) Transitive Property

If $a = b$ and $b = c$, then $a = c$, $\quad \forall \quad a, b, c \in R$

(iv) Additive Property

If $a = b$, then $a + c = b + c$, $\quad \forall \quad a, b, c \in R$

(v) Multiplicative Property

If $a = b$, then $ac = bc$, $\quad \forall \quad a, b, c \in R$

(vi) Cancellation Property for Addition

If $a + c = b + c$, then $a = b$, $\quad \forall \quad a, b, c \in R$

(vii) Cancellation Property for Multiplication

If $ac = bc$, $c \neq 0$ then $a = b$, $\quad \forall \quad a, b, c \in R$

(c) Properties of Inequalities of Real Numbers

Properties of inequalities of real numbers are as follows:

(i) Trichotomy Property

$\forall\ a, b \in R$

$a < b$ or $a = b$ or $a > b$

(ii) Transitive Property

$\forall\ a, b, c \in R$

(a) $a < b$ and $b < c \Rightarrow a < c$ (b) $a > b$ and $b > c \Rightarrow a > c$

(iii) Additive Property

$\forall\ a, b, c \in R$

$a < b \Rightarrow a + c < b + c$
$a < b \Rightarrow c + a < c + b$

and (b)

$a > b \Rightarrow a + c > b + c$
$a > b \Rightarrow c + a > c + b$

(iv) Multiplicative Property

(a) $\forall\ a, b, c \in R$ and $c > 0$

(i) $a > b \Rightarrow ac > bc$
$a > b \Rightarrow ca > cb$

(ii) $a < b \Rightarrow ac < bc$
$a < b \Rightarrow ca < cb$

(b) $\forall\ a, b, c \in R$ and $c < 0$

(i) $a > b \Rightarrow ac < bc$
$a > b \Rightarrow ca < cb$

(ii) $a < b \Rightarrow ac > bc$
$a < b \Rightarrow ca > cb$

(v) Multiplicative Inverse Property

$\forall\ a, b \in R$ and $a \neq 0, b \neq 0$

(a) $a < b \Leftrightarrow \frac{1}{a} > \frac{1}{b}$ (b) $a > b \Leftrightarrow \frac{1}{a} < \frac{1}{b}$

EXERCISE 2.2

1. Identify the property used in the following

(i)	$a + b = b + a$	(ii)	$(ab)c = a(bc)$
(iii)	$7 \times 1 = 7$	(iv)	$x > y$ or $x = y$ or $x < y$
(v)	$ab = ba$	(vi)	$a + c = b + c \Rightarrow a = b$
(vii)	$5 + (-5) = 0$	(viii)	$7 \times \frac{1}{7} = 1$
(ix)	$a > b \Rightarrow ac > bc \quad (c > 0)$		

2. Fill in the following blanks by stating the properties of real numbers used.

$3x + 3(y - x)$
$= 3x + 3y - 3x,$ ....
$= 3x - 3x + 3y,$ ....
$= 0 + 3y,$ ....
$= 3y$ ....

3. Give the name of property used in the following.

(i) $\sqrt{24} + 0 = \sqrt{24}$

(ii) $-\frac{2}{3}\left(5 + \frac{7}{2}\right) = \left(-\frac{2}{3}\right)(5) + \left(-\frac{2}{3}\right)\left(\frac{7}{2}\right)$

(iii) $\pi + (-\pi) = 0$

(iv) $\sqrt{3} \cdot \sqrt{3}$ is a real number

(v) $\left(-\frac{5}{8}\right)\left(-\frac{8}{5}\right) = 1$

2.3 Radicals and Radicands

2.3.1 Concept of Radicals and Radicands

If n is a positive integer greater than 1 and a is a real number, then any real number x such that $x^n = a$ is called the nth root of a, and in symbols is written as

$$x = \sqrt[n]{a}, \quad or \quad x = (a)^{1/n},$$

In the radical $\sqrt[n]{a}$, the symbol $\sqrt{\ }$ is called the radical sign, n is

called the index of the radical and the real number a under the radical sign is called the radicand or base.

Note:

$\sqrt[2]{a}$ is usually written as $\sqrt{a}$.

2.3.2 Difference between Radical form and Exponential form

In radical form, radical sign is used

e.g., $x = \sqrt[n]{a}$ is a radical form.

$\sqrt[3]{x}$, $\sqrt[5]{x^2}$ are examples of radical form.

In exponential form, exponential is used in place of radicals,

e.g., $x = (a)^{1/n}$ is exponential form.

$x^{3/2}$, $z^{2/7}$ are examples of exponential form.

Properties of Radicals

Let a, b d R and m, n be positive integers. Then,

(i) $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ (ii) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

(iii) $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$ (iv) $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$ (v) $\sqrt[n]{a^n} = a$

2.3.3 Transformation of an Expression given in Radical form to Exponential form and vice versa

The method of transforming expression in radical form to exponential form and vice versa is explained in the following examples.

Example 1

Write each radical expression in exponential notation and each exponential expression in radical notation. Do not simplify.

(i) $\sqrt[5]{-8}$ (ii) $\sqrt[3]{x^5}$ (iii) $y^{3/4}$ (iv) $x^{-3/2}$

Solution

(i) $\sqrt[5]{-8} = (-8)^{1/5}$ (ii) $\sqrt[3]{x^5} = x^{5/3}$

(iii) $y^{3/4} = \sqrt[4]{y^3}$ or $(\sqrt[4]{y})^3$ (iv) $x^{-3/2} = \sqrt{x^{-3}}$ or $(\sqrt{x})^{-3}$

Example 2

Simplify $\sqrt[3]{16x^4y^5}$

Solution

$$\sqrt[3]{16x^4y^5} = \sqrt[3]{(2)(8)(x)(x^3)(y^2)(y^3)}, \quad \text{...... (factorizing)}$$
$$= \sqrt[3]{2xy^2\,(2^3)(x^3)(y^3)}, \quad \text{...... (arranging perfect cubes)}$$
$$= \sqrt[3]{2xy^2}\,\sqrt[3]{(2^3)(x^3)(y^3)}, \quad \text{...... property (i)}$$
$$= \sqrt[3]{2xy^2}\,\sqrt[3]{2^3}\,\sqrt[3]{x^3}\,\sqrt[3]{y^3}, \quad \text{...... property (i)}$$
$$= 2xy\sqrt[3]{2xy^2}, \quad \text{...... property (v)}$$

EXERCISE 2.3

1. Write each radical expression in exponential notation and each exponential expression in radical notation. Do not simplify.

(i) $\sqrt[3]{-64}$ (ii) $2^{3/5}$ (iii) $-7^{1/3}$ (iv) $y^{-2/3}$

2. Tell whether the following statements are true or false?

(i) $5^{1/5} = \sqrt{5}$ (ii) $2^{2/3} = \sqrt[3]{4}$ (iii) $\sqrt{49} = \sqrt{7}$ (iv) $\sqrt[3]{x^{27}} = x^3$

3. Simplify the following radical expressions.

(i) $\sqrt[3]{-125}$ (ii) $\sqrt[4]{32}$ (iii) $\sqrt[5]{\frac{3}{32}}$ (iv) $\sqrt[3]{-\frac{8}{27}}$

2.4 Laws of Exponents / Indices

2.4.1 Base and Exponent

In the exponential notation a^n (read as a to the nth power) we call 'a' as the base and 'n' as the exponent or the power to which the base is raised.

From this definition, recall that, we have the following laws of exponents.

If $a, b \in$ R and m, n are positive integers, then

I	$a^m \cdot a^n = a^{m+n}$	II	$(a^m)^n = a^{mn}$
III	$(ab)^n = a^n b^n$	IV	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$
V	$= a^m/a^n, a^{m-n}, a \neq 0$	VI	$a^0 = 1$, where $a \neq 0$
VII	$a^{-n} = \frac{1}{a^n}$, where $a \neq 0$		

2.4.2 Applications of Laws of Exponents

The method of applying the laws of indices to simplify algebraic expressions is explained in the following examples.

Example 1

Use rules of exponents to simplify each expression and write the answer in terms of positive exponents.

(i) $\frac{x^{-2}x^{-3}y^7}{x^{-3}y^4}$ (ii) $\left(\frac{4a^3b^0}{9a^{-5}}\right)^{-2}$

Solution

(i) $\frac{x^{-2}x^{-3}y^7}{x^{-3}y^4} = \frac{x^{-5}y^7}{x^{-3}y^4}$ $(a^m a^n = a^{m+n})$

$= \frac{y^{7-4}}{x^{-3+5}} = \frac{y^3}{x^2}$ $\left(\frac{a^m}{a^n} = a^{m-n}\right)$

(ii) $\left(\frac{4a^3b^0}{9a^{-5}}\right)^{-2} = \left(\frac{4a^{3+5}\times 1}{9}\right)^{-2}$ $\left(\frac{a^m}{a^n} = a^{m-n}, b^0 = 1\right)$

$= \left(\frac{4a^8}{9}\right)^{-2} = \left(\frac{9}{4a^8}\right)^{+2}$ $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

$= \frac{81}{16a^{16}}$ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Example 2

Simplify the following by using laws of indices:

(i) $\left(\frac{8}{125}\right)^{-4/3}$ (ii) $\frac{4(3)^n}{3^{n+1} - 3^n}$

Solution

Using Laws of Indices,

(i) $\left(\frac{8}{125}\right)^{-4/3} = \left(\frac{125}{8}\right)^{4/3} = \frac{(125)^{4/3}}{(8)^{4/3}} = \frac{(5^3)^{4/3}}{(2^3)^{4/3}} = \frac{5^4}{2^4} = \frac{625}{16}$

(ii) $\frac{4(3)^n}{3^{n+1} - 3^n} = \frac{4(3)^n}{3^n[3-1]} = \frac{4(3)^n}{2(3^n)} = \frac{4}{2} = 2$

EXERCISE 2.4

1. Use laws of exponents to simplify:

(i) $\frac{(243)^{-2/3}(32)^{-1/5}}{\sqrt{(196)^{-1}}}$ (ii) $(2x^5y^{-4})(-8x^{-3}y^2)$

(iii) $\left(\dfrac{x^{-2}y^{-1}z^{-4}}{x^{4}y^{-3}z^{0}}\right)^{-4/3}$ (iv) $\dfrac{(81)^{n}.3^{5}-(3)^{4n-1}(243)}{(9^{2n})(3^{3})}$

2. Show that

$$\left(\frac{x^a}{x^b}\right)^{a+b}\times\left(\frac{x^b}{x^c}\right)^{b+c}\times\left(\frac{x^c}{x^a}\right)^{c+a}=1$$

3. Simplify

(i) $\dfrac{2^{1/3}\times(27)^{1/3}\times(60)^{1/2}}{(180)^{1/2}\times(4)^{-1/3}\times(9)^{1/4}}$ (ii) $\sqrt{\dfrac{(216)^{2/3}\times(25)^{1/2}}{(.04)^{-1/2}}}$

(iii) $5^{2^{3}}\div(5^{2})^{3}$ (iv) $(x^{3})^{2}\div x^{3^{2}},\ x\neq 0$

2.5 Complex Numbers

We recall that the square of a real number is non-negative. So the solution of the equation $x^2 + 1 = 0$ or $x^2 = -1$ does not exist in R. To overcome this inadequacy of real numbers, we need a number whose square is –1. Thus the mathematicians were tempted to introduce a larger set of numbers called the set of complex numbers which contains R and every number whose square is negative. They invented a new number –1, called the imaginary unit, and denoted it by the letter i (iota) having the property that $i^2=-1$. Obviously i is not a real number. It is a new mathematical entity that enables us to enlarge the number system to contain solution of every algebraic equation of the form $x^2 = -a$, where $a > 0$. By taking new number $i=\sqrt{-1}$, the solution set of $x^2 + 1 = 0$ is

$$\{\sqrt{-1},-\sqrt{-1}\}\quad\text{or}\quad\{i,-i\}$$

Note:

The Swiss mathematician Leonard Euler (1707 – 1783) was the first to use the symbol i for the number $\sqrt{-1}$
Numbers like $\sqrt{-1},\sqrt{-5}$ etc. are called pure imaginary numbers.

Integral Powers of i

By using $i=\sqrt{-1}$,we can easily calculate the integral powers of i.
e.g., $i^2=-1,\ i^3=i^2\times i=-i,\ i^4=i^2\times i^2=(-1)(-1)=1,\ i^8=(i^2)^4=(-1)^4=1,$
$i^{10}=(i^2)^5=(-1)^5=-1,$ etc.

A pure imaginary number is the square root of a negative real number.

2.5.1 Definition of a Complex Number

A number of the form z = a + bi where a and b are real numbers and $i=\sqrt{-1}$,is called a complex number and is represented by z i.e., z = a + ib

2.5.2 Set of Complex Numbers

The set of all complex numbers is denoted by C, and
$C=\{z\,|\,z=a+bi$, where $a,b\in R$ and $i=\sqrt{-1}\}$

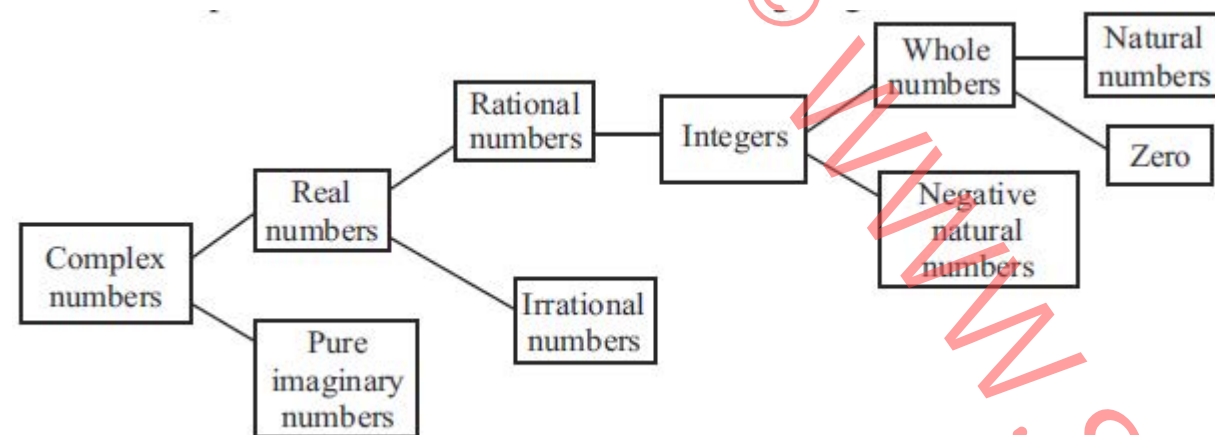
The numbers *a* and *b*, called the real and imaginary parts of *z*, are denoted as *a* = R(*z*) and *b* = Im(*z*).

Observe that:

(i) Every a ∈ R may be identified with complex numbers of the form a + Oi taking b = 0. Therefore, every real number is also a complex number. Thus R ⊂ C. Note that every complex number is not a real number.

(ii) If a = 0, then a + bi reduces to a purely imaginary number bi. The set of purely imaginary numbers is also contained in C.

(iii) If a= b = 0, then z = 0 + i0 is called the complex number 0.

The set of complex numbers is shown in the following diagram

2.5.3 Conjugate of a Complex Number

If we change i to –i in z = a + bi, we obtain another complex number a – bi called the complex conjugate of z and is denoted by $\bar{z}$ (read z bar).

Thus, if $z = -1 - i$, then $\bar{z} = -1 + i$.

The numbers a + bi and a — bi are called conjugates of each other.

Note that:

(i) $\bar{\bar{z}} = z$

(ii) The conjugate of a real number $z = a + 0i$ coincides with the number itself, since $\bar{z} = \overline{a + 0i} = a - 0i$.

(iii) conjugate of a real number is the same real number.

2.5.4 Equality of Complex Numbers and its Properties

For all a, b, c, d ∈ R,

a + bi = c + di if and only if a= c and b = d.

e.g., $2x + y^2 i = 4 + 9i$ if and only if

$2x = 4$ and $y^2 = 9$, i.e., $x = 2$ and $y = \pm 3$

Properties of real numbers R are also valid for the set of complex numbers.

(i) $z_1 = z_1$ (Reflexive law)

(ii) If $z_1 = z_2$, then $z_2 = z_1$ (Symmetric law)

(iii) If $z_1 = z_2$ and $z_2 = z_3$, then $z_1 = z_3$ (Transitive law)

EXERCISE 2.5

1. Evaluate

(i) i^7 (ii) i^{50} (iii) i^{12}

(iv) $(-)^8$ (v) $(-i)^5$ (vi) i^{27}

2. Write the conjugate of the following numbers.

(i) 2 + 3i (ii) 3 – 5i (iii) –i

(iv) –3 + 4i (v) –4 – i (vi) i – 3

3. Write the real and imaginary part of the following numbers.

(i) 1 + i (ii) –1 + 2i (iii) –3i + 2

(iv) –2 – 2i (v) –3i (vi) 2 + 0i

4. Find the value of x and y if x + iy + 1 = 4 – 3i.

2.6 Basic Operations on Complex Numbers

(i) **Addition**

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers and a, b, c, d ∈ R.

The sum of two complex numbers is given by

$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$

i.e., the sum of two complex numbers is the sum of the corresponding real and the imaginary parts.

e.g., (3 –8i) + (5 + 2i) = (3 + 5) + (–8 + 2)i = 8 – 6i

(i) Multiplication

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers.

The products are found as

(i) If $k \in R$, $kz_1 = k(a + bi) = ka + kbi$.

(Multiplication of a complex number with a scalar)

(ii) $z_1z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$

(Multiplication of two complex numbers)

The multiplication of any two complex numbers $(a + bi)$ and $(c + di)$ is explained as

$$z_1z_2 = (a + bi)(c + di) = a(c + di) + bi(c + di)$$
$$= ac + adi + bci + bdi^2$$
$$= ac + adi + bci + bd(-1) \quad \text{(since } i^2 = -1\text{)}$$
$$= (ac - bd) + (ad + bc)i \quad \text{(combining like terms)}$$

e.g., $(2 - 3i)(4 + 5i) = 8 + 10i - 12i - 15i^2 = 23 - 2i$. (since $i^2 = -1$)

(iii) Subtraction

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers.

The difference between two complex numbers is given by

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i$$

e.g., $(-2 + 3i) - (2 + i) = (-2 - 2) + (3 - 1)i = -4 + 2i$

i.e., the difference of two complex numbers is the difference of the corresponding real and imaginary parts.

(iv) Division

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers such that $z_2 \neq 0$.

The division of $a + bi$ by $c + di$ is given by

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di},$$ (Multiplying the numerator and denominator by $c - di$, the complex conjugateof $c + di$).

$$= \frac{ac + bci - adi - bdi^2}{c^2 - (di)^2}$$

$$= \frac{ac + bci - adi + bd}{c^2 + d^2}, \text{ since } i^2 = -1$$

$$= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)i$$

Operations are explained with the help of following examples.

Example 1

Separate the real and imaginary parts of $(-1 + \sqrt{-2})^2$

Solution

Let $z = -1 + \sqrt{-2}$, then

$$z^2 = (-1 + \sqrt{-2})^2 = (-1 + i\sqrt{2})^2, \text{ changing to } i\text{-form}$$
$$= (-1 + i\sqrt{2})(-1 + i\sqrt{2}) = (-1)(-1 + i\sqrt{2}) + i\sqrt{2}(-1 + i\sqrt{2})$$
$$= 1 - i\sqrt{2} - i\sqrt{2} + 2i^2 = -1 - 2\sqrt{2}\,i$$

Hence $\text{Re}(z^2) = -1$ and $\text{Im}(z^2) = -2\sqrt{2}$

Example 2

Express $\frac{1}{1+2i}$ in the standard form $a + bi$.

Solution

We have $\frac{1}{1+2i} = \frac{1}{1+2i} \times \frac{1-2i}{1-2i}$

(multiplying the numerator and denominator by $\overline{1 + 2i}$)

$$= \frac{1-2i}{1-(2i)^2} = \frac{1-2i}{1-4i^2}, \quad \text{(simplifying)}$$

$$= \frac{1-2i}{5}, \quad \text{(since } i^2 = -1\text{)}$$

$$= \frac{1}{5} - \frac{2}{5}i, \quad \text{which is of the form } a + bi$$

Example 3

Express $\frac{4+5i}{4-5i}$ in the standard form $a + bi$.

Solution

$\frac{4+5i}{4-5i}=(4+5i).\frac{1}{4-5i}\times\frac{4+5i}{4+5i}$ (multiplying and dividing by the conjugate of $(4-5)i$)

$=\frac{(4+5i)^2}{(4)^2-(5i)^2}=\frac{16+40i+25i^2}{16-25i^2}$ (simplifying)

$=\frac{16+40i+25}{16-25}$, (since $i^2=-1$)

$=\frac{-9+40i}{41}=-\frac{9}{41}+\frac{40}{41}i$

Example 4

Solve (3 – 4i) (x + yi) = 1 + 0 . i for real numbers x and y, where $i=\sqrt{-1}$.

Solution

We have (3 – 4i) (x + yi) = 1 + 0 . i
or $3x$ + 3iy – 4ix – 4i^{2}y = 1 + 0 . i
or $3x$ + 4y + (3y – 4x)i = 1 + 0 . i
Equating the real and imaginary parts, we obtain
$3x$ + 4y = 1 and 3y – 4x = 0
olving these two equations simultaneously, we have $x=\frac{3}{25}$ and $y=\frac{4}{25}$

EXERCISE 2.6

1. Identify the following statements as true or false.
   (i) $\sqrt{-3}\sqrt{-3}=3$ (ii) $i^{73}=-i$ (iii) $i^{10}=-1$
   (iv) Complex conjugate of $(-6i+i^2)$ is $(-1+6i)$
   (v) Difference of a complex number z = a + bi and its conjugate is a real number.
   (vi) If (a – 1) – (b +3)i = 5 + 8i, then a = 6 and b = –11.
   (vii) Product of a complex number and its conjugate is always a non-negative real number.
2. Express each complex number in the standard form a + bi, where a and b are real numbers.
   (i) (2 + 3i) + (7 - 2i) (ii) 2(5 + 4i) -3 (7 + 4i)
   (iii) -(-3 + 5i) - (4 + 9i) (iv) $2i^2+6i^3+3i^{16}-6i^{19}+4i^{25}$
3. Simplify and write your answer in the form a+ bi.
   (i) $(-7+3i)(-3+2i)$ (ii) $(2-\sqrt{-4})(3-\sqrt{-4})$
   (iii) $(\sqrt{5}-3i)^2$ (iv) $(2-3i)(\overline{3-2i})$
4. Simplify and write your answer in the form a + bi.
   (i) $\frac{-2}{1+i}$ (ii) $\frac{2+3i}{4-i}$ (iii) $\frac{9-7i}{3+i}$
   (iv) $\frac{2-6i}{3+i}-\frac{4+i}{3+i}$ (v) $\left(\frac{1+i}{1-i}\right)^2$ (vi) $\frac{1}{(2+3i)(1-i)}$
5. Calculate (a) $\bar{z}$ (b) $z+\bar{z}$ (c) $z-\bar{z}$ (d) $z\bar{z}$, for each of the following
   (i) z = -i (ii) z = 2 + i
   (iii) $z=\frac{1+i}{1-i}$ (iv) $z=\frac{4-3i}{2+4i}$
6. If z = 2 + 3i and w = 5 - 4i, show that
   (i) $\overline{z+w}=\bar{z}+\bar{w}$ (ii) $\overline{z-w}=\bar{z}-\bar{w}$
   (iii) $\overline{zw}=\bar{z}\,\bar{w}$ (iv) $\overline{\left(\frac{z}{w}\right)}=\frac{\bar{z}}{\bar{w}}$, where $w\neq 0$.
   (v) $\frac{1}{2}(z+\bar{z})$ is the real part of z. (vi) $\frac{1}{2i}(z-\bar{z})$ is the imaginary part of z.
7. Solve the following equations for real x and y.
   (i) (2 – 3i) (x + yi) = 4 + i
   (ii) (3 – 2i) (x + yi) = 2(x – 2yi) + 2i – 1
   (iii) $(3+4i)^2-2(x-yi)=x+yi$

REVIEW EXERCISE 2

1. **Multiple Choice Questions. Choose the correct answer.**

2. **True or false? Identify.**
   (i) Division is not an associative operation. ....
   (ii) Every whole number is a natural number. ....
   (iii) Multiplicative inverse of 0.02 is 50. ....
   (iv) π is a rational number. ....
   (v) Every integer is a rational number. ....
   (vi) Subtraction is a commutative operation. ....
   (vii) Every real number is a rational number. ....
   (viii) Decimal representation of a rational number is either terminating or recurring. ....

3. Simplify the following:

   (i) $\sqrt[4]{81y^{-12}x^{-8}}$ (ii) $\sqrt{25x^{10n}y^{8m}}$

   (iii) $\left(\dfrac{x^3y^4z^5}{x^{-2}y^{-1}z^{-5}}\right)^{1/5}$ (iv) $\left(\dfrac{32x^{-6}y^{-4}z}{625x^4y\,z^{-4}}\right)^{2/5}$

4. Simplify $\sqrt{\dfrac{(216)^{2/3}\times(25)^{1/2}}{(0.04)^{-3/2}}}$

5. Simplify

   $\left(\dfrac{a^p}{a^q}\right)^{p+q}\cdot\left(\dfrac{a^q}{a^r}\right)^{q+r}\div 5(a^p\,.\,a^r)^{p-r}, a\neq 0$

6. Simplify $\left(\dfrac{a^{2l}}{a^{l+m}}\right)\left(\dfrac{a^{2m}}{a^{m+n}}\right)\left(\dfrac{a^{2n}}{a^{n+l}}\right)$

7. Simplify $\sqrt[3]{\dfrac{a^l}{a^m}}\times\sqrt[3]{\dfrac{a^m}{a^n}}\times\sqrt[3]{\dfrac{a^n}{a^l}}$

SUMMARY

* Set of real numbers is expressed as R = Q U Q′ where

  $Q=\left\{\dfrac{p}{q}\,|\,p,q\in Z\wedge q\neq 0\right\}$, $Q'=\{x\,|\,x$ is not rational$\}$.

* Properties of real numbers w.r.t. addition and multiplication:

  Closure: $a+b\in R$, $ab\in R$, $\forall$ $a, b\in R$

  Associative:

  $(a+b)+c=a+(b+c)$, $(ab)c=a(bc)$, $\forall$ $a, b, c\in R$

  Commutative:

  $a+b=b+a$, $ab=ba$, $\forall$ $a, b\in R$

  Additive Identity:

  $a+0=a=0+a$, $\forall$ $a\in R$

  Multiplicative Identity:

  $a\,.\,1=a=1\,.\,a$, $\forall$ $a\in R$

  Additive Inverse:

  $a+(-a)=0=(-a)+a$, $\forall$ $a\in R$

  Multiplicative Inverse:

  $a\cdot\dfrac{1}{a}=1=\dfrac{1}{a}\cdot a$, $a\neq 0$

  Multiplication is distributive over addition and subtraction:

  $a(b+c)=ab+ac$, $\forall$ $a, b, c\in R$
  $(b+c)a=ba+ca$, $\forall$ $a, b, c\in R$
  $a(b-c)=ab-ac$, $\forall$ $a, b, c\in R$
  $(a-b)c=ac-bc$, $\forall$ $a, b, c\in R$

- Properties of equality in R
  Reflexive: $a = a, \ \forall \ a \in R$
  Symmetric: $a = b \Rightarrow b = a, \ \forall \ a, b \in R$
  Transitive: $a = b, b = c \Rightarrow a = c, \ \forall \ a, b, c \in R$
  Additive property: If a = b, then $a + c = b + c, \ \forall \ a, b, c \in R$
  Multiplicative property: If $a = b$, then $ac = bc, \ \forall \ a, b, c \in R$
  Cancellation property: If $ac = be$, $c \neq 0$, then $a = b, \forall \ a, b, c \in R$
- In the radical $\sqrt[n]{x}$, $\sqrt{\ }$ is radical sign, x is radicand or base and n is index of radical.
- Indices and laws of indices:
  $\forall a, b, c \in R$ and $m, n \in z$,
  $(a^m)^n = a^{mn}$, $(ab)^n = a^n b^n$

  $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$$

  $a^m a^n = a^{m+n}$

  $$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$$

  $$a^{-n} = \frac{1}{a^n}, a \neq 0$$

  $$a^0 = 1$$

- Complex number $z = a + bi$ is defined using imaginary unit $i = \sqrt{-1}$.
  where $a, b \in R$ and $a = \text{Re}(z)$, $b = \text{Im}(z)$
- Conjugate of $z = a + bi$ is defined as $\bar{z} = a - bi$

CHAPTER 3

LOGARITHMS

Animation 3.1:Laws of logarithms
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Students Learning Outcomes

After studying this unit, the students will be able to:

- express a number in standard form of scientific notation and vice versa.
- define logarithm of a number y to the base a as the power to which a must be raised to give the number (i.e., $a^x = y \Leftrightarrow \log_a y = x$, $a > 0$,
- $a \neq 1$ and $y > 0$).
- define a common logarithm, characteristic and mantissa of log of a number.
- use tables to find the log of a number.
- give concept of antilog and use tables to find the antilog of a number.
- differentiate between common and natural logarithm.
- prove the following laws of logarithm
  - $\log_a(mn) = \log_a m + \log_a n$,
  - $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$,
  - $\log_a m^n = n\log_a m$,
  - $\log_a m \log_m n = \log_a n$.
- apply laws of logarithm to convert lengthy processes of multiplication, division and exponentiation into easier processes of addition and subtraction etc.

Introduction

The difficult and complicated calculations become easier by using logarithms.

Abu Muhammad Musa Al Khwarizmi first gave the idea of logarithms. Later on, in the seventeenth century John Napier extended his work on logarithms and prepared tables for logarithms He used "e" as the base for the preparation of logarithm tables. Professor Henry Briggs had a special interest in the work of John Napier. He prepared logarithim tables with base 10. Antilogarithm table was prepared by Jobst Burgi in 1620 A.D.

3.1 Scientific Notation

There are so many numbers that we use in science and technical work that are either very small or very large. For instance, the distance from the Earth to the Sun is 150,000,000 km approximately and a hydrogen atom weighs 0.000,000,000,000,000,000,000,001,7 gram. While writing these numbers in ordinary notation (standard notation) there is always chance of making an error by omitting a zero or writing more than actual number of zeros. To overcome this problem, scientists have developed a concise, precise and convenient method to write very small or very large numbers, that is called scientific notation of expressing an ordinary number.

A number written in the form $a \times 10^n$, where $1 \leq a < 10$ and n is an integer, is called the scientific notation.

The above mentioned numbers (in 3.1) can be conveniently written in scientific notation as 1.5×10^8 km and 1.7×10^{-24} gm respectively.

Example 1

Write each of the following ordinary numbers in scientific notation

(i) 30600 (ii) 0.000058

Solution

$30600 = 3.06 \times 10^4$ (move decimal point four places to the left)

$0.000058 = 5.8 \times 10^{-5}$ (move decimal point five places to the right)

Observe that for expressing a number in scientific notation

(i) Place the decimal point after the first non-zero digit of given number.

(ii) We multiply the number obtained in step (i), by 10^n if we shifted the decimal point n places to the left

(iii) We multiply the number obtained in step (i) by 10^{-n} if we shifted the decimal point n places to the right.

(iv) On the other hand, if we want to change a number from scientific notation to ordinary (standard) notation, we simply reverse the process.

Example 2

Change each of the following numbers from scientific notation to ordinary notation. (i) 6.35×10^6 (ii) 7.61×10^{-4}

Solution

(i) $6.35 \times 10^6 = 6350000$ (move the decimal point six places to the right)

(ii) $7.61 \times 10^{-4} = 0.000761$ (move the decimal point four places to the left)

EXERCISE 3.1

Express each of the following numbers in scientific notation.

(i) 5700 (ii) 49,800,000 (iii) 96,000,000
(iv) 416.9 (v) 83,000 (vi) 0.00643
(vii) 0.0074 (viii) 60,000,000 (ix) 0.00000000395
(x) $\frac{275,000}{0.0025}$

Express the following numbers in ordinary notation.

(i) 6×10^{-4} (ii) 5.06×10^{10}
(iii) 9.018×10^{-6} (iv) 7.865×10^8

3.2 Logarithm

Logarithms are useful tools for accurate and rapid computations. Logarithms with base 10 are known as common logarithms and those with base e are known as natural logarithms. We shall define logarithms with base $a > 0$ and $a \neq 1$.

3.2.1 Logarithm of a Real Number

If $a^x = y$, then x is called the logarithm of y to the base 'a' and is written as

$\log_a y = x$, where $a > 0$, $a \neq 1$ and $y > 0$.

i.e., the logarithm of a number y to the base 'a' is the index x of the power to which a must be raised to get that number y.

The relations $a^x = y$ and $\log_a y = x$ are equivalent. When one relation is given, it can be converted into the other. Thus

$$a^x = y \Leftrightarrow \log_a y = x$$

$a^x = y$ and $\log_a y = x$ are respectively exponential and logarithmic form of the same relation.

To explain these remarks ,we observe that

$3^2 = 9$ is equivalent to $\log_3 9 = 2$

and $2^{-1} = \frac{1}{2}$ is equivalent to $\log_2 \left(\frac{1}{2}\right) = -1$

Logarithm of a negative number is not defined at this stage.

Similarly, we can say that

$\log_3 27 = 3$ is equivalent to $27 = 3^3$

Example 3

Find $\log_4 2$, i.e., find log of 2 to the base 4.

Solution

Let $\log_4 2 = x$.

Then its exponential form is $4^x = 2$

i.e., $2^{2x} = 2^1 \Rightarrow 2x = 1$

$\therefore x = \frac{1}{2} \Rightarrow \log_4 2 = \frac{1}{2}$

Deductions from Definition of Logarithm

1. Since $a^0 = 1$, $\log_a 1 = 0$
2. Since $a^1 = a$, $\log_a a = 1$

3.2.2 Definitions of Common Logarithm, Characteristic and Mantissa Definition of Common Logarithm

In numerical calculations, the base of logarithm is always taken as 10. These logarithms are called common logarithms or Briggesian logarithms in honour of Henry Briggs, an English mathematician and astronomer, who developed them.

Characteristic and Mantissa of Log of a Number

Consider the following

$10^3 = 1000 \Leftrightarrow \log 1000 = 3$

$10^2 = 100 \Leftrightarrow \log 100 = 2$

$10^1 = 10 \Leftrightarrow \log 10 = 1$

$10^0 = 1 \Leftrightarrow \log 1 = 0$

$10^{-1} = 0.1 \Leftrightarrow \log 0.1 = -1$

$10^{-2} = 0.01 \Leftrightarrow \log 0.01 = -2$

$10^{-3} = 0.001 \Leftrightarrow \log 0.001 = -3$

Note:

By convention, if only the common logarithms are used throughout a discussion, the base 10 is not written.

Also consider the following table

For the numbers	the logarithm is
Between 1 and 10	a decimal
Between 10 and 100	1 + a decimal
Between 100 and 1000	2 + a decimal
Between 0.1 and 1	–1 + a decimal
Between 0.01 and 0.1	–2 + a decimal
Between 0.001 and 0.01	–3 + a decimal

Observe that

The logarithm of any number consists of two parts:

(i) An integral part which is positive for a number greater than 1 and negative for a number less than 1, is called the characteristic of logarithm of the number.

(ii) A decimal part which is always positive, is called the mantissa of the logarithm of the number.

(i) Characteristic of Logarithm of a Number > 1

The first part of above table shows that if a number has one digit in the integral part, then the characteristic is zero; if its integral part has two digits, then the characteristic is one; with three digits in the integral part, the characteristic is two, and so on.

In other words, the characteristic of the logarithm of a number greater than 1 is always one less than the number of digits in the integral part of the number.

When a number b is written in the scientific notation, i.e., in the form $b = a \times 10^n$ where $1 \leq a < 10$, the power of 10 i.e., n will give the characteristic of log b.

Examples

Number	Scientific Notation	Characteristic of the Logarithm
1.02	1.02×10^0	0
99.6	9.96×10^1	1
102	1.02×10^2	2
1662.4	1.6624×10^3	3

Characteristic of Logarithm of a Number < 1

The second part of the table indicates that, if a number has no zero immediately after the decimal point, the characteristic is –1; if it has one zero immediately after the decimal point, the characteristic is –2; if it has two zeros immediately after the decimal point, the characteristic is –3; etc.

In other words, the characteristic of the logarithm of a number less than 1, is always negative and one more than the number of zeros immediately after the decimal point of the number.

Example

Write the characteristic of the log of following numbers by expressing them in scientific notation and noting the power of 10.

0.872, 0.02, 0.00345

Solution

Number	Scientific Notation	Characteristic of the Logarithm
0.872	8.72×10^{-1}	–1
0.02	2.0×10^{-2}	–2
0.00345	3.45×10^{-3}	–3

When a number is less than 1, the characteristic of its logarithm is written by convention, as $\bar{3}$, $\bar{2}$ or $\bar{1}$ instead of -3, -2 or -1 respectively ($\bar{3}$ is read as bar 3) to avoid the mantissa becoming negative.

Note:

$\bar{2}.3748$ does not mean –2.3748. In $\bar{2}.3748$, 2 is negative but .3748 is positive; Whereas in –2.3748 both 2 and .3748 are negative.

(ii) Finding the Mantissa of the Logarithm of a Number

While the characteristic of the logarithm of a number is written merely by inspection, the mantissa is found by making use of

logarithmic tables. These tables have been constructed to obtain the logarithms up to 7 decimal places. For all practical purposes, a four-figure logarithmic table will provide sufficient accuracy.

A logarithmic table is divided into 3 parts.

(a) The first part of the table is the extreme left column headed by blank square. This column contains numbers from 10 to 99 corresponding to the first two digits of the number whose logarithm is required.

(b) The second part of the table consists of 10 columns, headed by 0, 1, 2, ...,9. These headings correspond to the third digit from the left of the number. The numbers under these columns record mantissa of the logarithms with decimal point omitted for simplicity.

(c) The third part of the table further consists of small columns known as mean differences columns headed by 1, 2, 3, ...,9. These headings correspond to the fourth digit from the left of the number. The readings of these columns are added to the mantissa recorded in second part (*b*) above.

When the four-figure log table is used to find the mantissa of the logarithm of a number, the decimal point is ignored and the number is rounded to four significant figures.

3.2.3 Using Tables to find log of a Number

The method to find log of a number is explained in the following examples. In the first two examples, we shall confine to finding mantissa only.

Example 1

Find the mantissa of the logarithm of 43.254

Solution

Rounding off 43.254 we consider only the four significant digits 4325

1. We first locate the row corresponding to 43 in the log tables and
2. Proceed horizontally till we reach the column corresponding to 2. The number at the intersection is 6355.
3. Again proceeding horizontally till the mean difference column corresponding to 5 intersects this row, we get the number 5 at the intersection.
4. Adding the two numbers 6355 and 5, we get .6360 as the mantissa of the logarithm of 43.25.

Example 2

Find the mantissa of the logarithm of 0.002347

Solution

Here also, we consider only the four significant digits 2347
We first locate the row corresponding to 23 in the logarithm tables and proceed as before.
Along the same row to its intersection with the column corresponding to 4 the resulting number is 3692. The number at the intersection of this row and the mean difference column corresponding to 7 is 13. Hence the sum of 3692 and 13 gives the mantissa of the logarithm of 0.002347 as 0.3705

Note:

> The logarithms of numbers having the same sequence of significant digits have the same mantissa. e.g., the mantissa of log of numbers 0.002347 and 0.2347 is 0.3705

For finding the common logarithm of any given number,

(i) Round off the number to four significant digits.
(ii) Find the characteristic of the logarithm of the number by inspection.
(iii) Find the mantissa of the logarithm of the number from the log tables.
(iv) Combine the two.

Example 3

Find (i) log 278.23 (ii) log 0.07058

Solution

(i) 278.23 can be round off as 278.2
The characteristic is 2 and the mantissa, using log tables, is .4443
$\therefore$ log 278.23 = 2.4443
(ii) The characteristic of log 0.07058 is –2 which is written as $\overline{2}$ by convention. Using log tables the mantissa is .8487, so that
log 0.07058 = $\overline{2}.8487$

3.2.4 The Concept of Antilogarithm and Use of Antilog Tables

The number whose logarithm is given is called antilogarithm.
i.e., if $\log_a y = x$, then y is the antilogarithm of x, or y = antilog x

Finding the Number whose Logarithm is Known

We ignore the characteristic and consider only the mantissa. In the antilogarithm page of the log table, we locate the row corresponding to the first two digits of the mantissa (taken together with the decimal point). Then we proceed along this row till it intersects the column corresponding to the third digit of the mantissa. The number at the intersection is added with the number at the intersection of this row and the mean difference column corresponding to the fourth digit of the mantissa.

Thus the significant figures of the required number are obtained. Now only the decimal point is to be fixed.

(i) If the characteristic of the given logarithm is positive, that number increased by 1 gives the number of figures to the left of the decimal point in the required number.
(ii) If the characteristic is negative, its numerical value decreased by 1 gives the number of zeros to the right of the decimal point in the required number.

Example

Find the numbers whose logarithms are (i) 1.3247 (ii) $\bar{2}.1324$

Solution

(i) 1.3247

Reading along the row corresponding to .32 (as mantissa = 0.3247), we get 2109 at the intersection of this row with the column corresponding to 4. The number at the intersection of this row and the mean difference column corresponding to 7 is 3. Adding 2109 and 3 we get 2112.

Since the characteristic is 1 it is increased by 1 (because there should be two digits in the integral part) and therefore the decimal point is fixed after two digits from left in 2112.

Hence antilog of 1.3247 is 21.12.

(ii) $\bar{2}$.1324

Proceeding as in (i) the significant figures corresponding to the mantissa 0.1324 are 1356. Since the characteristic is $\bar{2}$, its numerical value 2 is decreased by 1. Hence there will be one zero after the decimal point.

Hence antilog of $\bar{2}.1324$ is 0.01356.

EXERCISE 3.2

1. Find the common logarithm of each of the following numbers.
   (i) 232.92 (ii) 29.326
   (iii) 0.00032 (iv) 0.3206
2. If log 31.09 = 1.4926, find values of the following
   (i) log 3.109, (ii) log 310.9, (iii) log 0.003109,
   (iv) log 0.3109 without using tables.
3. Find the numbers whose common logarithms are
   (i) 3.5621 (ii) $\bar{1}.7427$
4. What replacement for the unknown in each of following will make the statement true?
   (i) $\log_3 81 = L$ (ii) $\log_a 6 = 0.5$
   (iii) $\log_5 n = 2$ (iv) $10^p = 40$
5. Evaluate
   (i) $\log_2 \frac{1}{128}$ (ii) log 512 to the base $2\sqrt{2}$.
6. Find the value of x from the following statements.
   (i) $\log_2 x = 5$ (ii) $\log_{81} 9 = x$ (iii) $\log_{64} 8 = \frac{x}{2}$
   (iv) $\log_x 64 = 2$ (v) $\log_3 x = 4$

3.3 Common Logarithm and Natural Logarithm

In 3.2.2 we have introduced common logarithm having base 10. Common logarithm is also known as decadic logarithms named after its base 10. We usually take log$x$ to mean $\log_{10} x$, and this type of logarithm is more convenient to use in numerical calculations. John Napier prepared the logarithms tables to the base e. Napier's logarithms are also called Natural Logarithms He released the first ever log tables in 1614. $\log_e x$ is conventionally given the notation In x.

In many theoretical investigations in science and engineering, it is often convenient to have a base e, an irrational number, whose value is 2.7182818...

3.4 Laws of Logarithm

In this section we shall prove the laws of logarithm and then apply them to find products, quotients, powers and roots of numbers.

(i) $\log_a(mn) = \log_a m + \log_a n$

(ii) $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iii) $\log_a m^n = n \log_a m$

(iv) $\log_a n = \log_b n \times \log_a b$

(i) $\log_a(mn) = \log_a m + \log_a n$

Proof

Let $\log_a m = x$ and $\log_a n = y$

Writing in exponential form $a^x = m$ and $a^y = n$.

$\therefore \quad a^x \times a^y = mn$

i.e., $a^{x+y} = mn$

or $\log_a(mn) = x + y = \log_a m + \log_a n$

Hence $\log_a(mn) = \log_a m + \log_a n$

Note:

(i) $\log_a(mn) \neq \log_a m \times \log_a n$

(ii) $\log_a m + \log_a n \neq \log_a(m + n)$

(iii) $\log_a(mnp \ldots) = \log_a m + \log_a n + \log_a p + \ldots$

The rule given above is useful in finding the product of two or more numbers using logarithms. We illustrate this with the following examples.

Example 1

Evaluate 291.3×42.36

Solution

Note that $\log_a a = 1$

Let $x = 291.3 \times 42.36$

Then $\log x = \log(291.3 \times 42.36)$

$= \log 291.3 + \log 42.36$, $\quad (\log_a mn = \log_a m + \log_a n)$

$= 2.4643 + 1.6269 = 4.0912$

$x = \text{antilog } 4.0912 = 12340$

Example 2

Evaluate 0.2913×0.004236.

Solution

Let $y = 0.2913 \times 0.004236$

Then $\log y = \log 0.2913 + \log 0.004236$

$= \bar{1}.4643 + \bar{3}.6269$

$= \bar{3}.0912$

Hence $y = \text{antilog } \bar{3}.0912 = 0.001234$

(ii) $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

Proof

Let $\log_a m = x$ and $\log_a n = y$

Then $a^x = m$ and $a^y = n$

$$\therefore \quad \frac{a^x}{a^y} = \frac{m}{n} \Rightarrow a^{x-y} = \frac{m}{n}$$

$$\text{i.e.,} \quad \log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n$$

$$\text{Hence } \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Note:

(i) $\log_a\left(\dfrac{m}{n}\right) \neq \dfrac{\log_a m}{\log_a n}$

(ii) $\log_a m - \log_a n \neq \log_a(m - n)$

(iii) $\log_a\left(\dfrac{1}{n}\right) = \log_a 1 - \log_a n = -\log_a n \qquad (\because \log_a 1 = 0)$

Example 1

Evaluate $\dfrac{291.3}{42.36}$

Solution

Let $x = \dfrac{291.3}{42.36}$, then $\log x = \log \dfrac{291.3}{42.36}$.

Then $\log x = \log 291.3 - \log 42.36 \qquad (\because \log_a \frac{m}{n} = \log_a m - \log_a n)$

$= 2.4643 - 1.6269 = 0.8374$

Thus $x = \text{antilog } 0.8374 = 6.877$

Example 2

Evaluate $\dfrac{0.002913}{0.04236}$

Solution

Let $y = \dfrac{0.002913}{0.04236}$, then $\log y = \log\left(\dfrac{0.002913}{0.04236}\right)$

or $\log y = \log 0.002913 - \log 0.04236$

$\log y = \bar{3}.4643 - \bar{2}.6269$

$= \bar{3} + (0.4643 - 0.6269) - \bar{2}$

$= \bar{3} - 0.1626 - \bar{2}$

$= \bar{3} + (1 - 0.1626) - 1 - \bar{2}$, (adding and subtracting 1)

$= \bar{2}.8374$ $[\because -3 - 1 - 2 = -3 - 1 - (-2) = -2 = \bar{2}]$

Therefore, $y = \text{antilog}\ \bar{2}.8374 = 0.06877$

(iii) $\log_a(m^n) = n \log_a m$

Proof

Let $\log_a m^n = x$, i.e., $a^x = m^n$

and $\log_a m = y$, i.e., $a^y = m$

Then $a^x = m^n = (a^y)^n$

i.e., $a^x = (a^y)^n = a^{yn} \Rightarrow x = ny$

i.e., $\log_a m^n = n \log_a m$

Example 1

Evaluate $\sqrt[4]{(0.0163)^3}$

Solution

Let $y = \sqrt[4]{(0.0163)^3} = (0.0163)^{3/4}$

Then $\log y = \dfrac{3}{4}(\log 0.0163) = \dfrac{3}{4} \times \bar{2}.2122 = \dfrac{\bar{6}.6366}{4} = \dfrac{\bar{8} + 2.6366}{4}$

$= \bar{2} + 0.6592 = \bar{2}.6592$

Hence $y = \text{antilog}\ \bar{2}.6592$

$= 0.04562$

(iv) Change of Base Formula

$$\log_a n = \log_b n \times \log_a b \quad \text{or} \quad \frac{\log_b n}{\log_b a}$$

Proof

Let $\log_b n = x$ so that $n = b^x$

Taking log to the base a, we have

$\log_a n = \log_a b^x = x \log_a b = \log_b n \log_a b$

Thus $\log_a n = \log_b n \log_a b$ ...... (i)

Putting $n = a$ in the above result, we get

$\log_b a \times \log_a b = \log_a a = 1$

or $\log_a b = \dfrac{1}{\log_b a}$

Hence equation (i) gives

$\log_a n = \dfrac{\log_b n}{\log_b a}$

Using the above rule, a natural logarithm can be converted to a common logarithm and vice versa.

$$\log_e n = \log_{10} n \times \log_e 10 \quad \text{or} \quad \frac{\log_{10} n}{\log_{10} e}$$

$$\log_{10} n = \log_e n \times \log_{10} e \quad \text{or} \quad \frac{\log_e n}{\log_e 10}$$

The values of $\log_e 10$ and $\log_{10} e$ are available from the tables:

$\log_e 10 = \dfrac{1}{0.4343} = 2.3026$ and $\log_{10} e = \log 2.718 = 0.4343$

Example:

Calculate $\log_2 3 \times \log_3 8$

Solution:

We know that

$\log_a n = \dfrac{\log_b n}{\log_b a}$

$$\therefore \quad \log_2 3 \times \log_3 8 = \frac{\log 3}{\log 2} \times \frac{\log 8}{\log 3} = \frac{\log 8}{\log 2} = \frac{\log 2^3}{\log 2} = \frac{3 \log 2}{\log 2} = 3$$

Note:

(i) During conversion the product form of the change of base rule may often be convenient.
(ii) Logarithms can be defined to any positive base other than 1, e or 10, and are useful for solving equations in which the unknown appears as the exponent of some other quantity.

EXERCISE 3.3

1. Write the following into sum or difference

(i) $\log (A \times B)$ (ii) $\log \frac{15.2}{30.5}$ (iii) $\log \frac{21 \times 5}{8}$

(iv) $\log \sqrt[3]{\frac{7}{15}}$ (v) $\log \frac{(22)^{1/3}}{5^3}$ (vi) $\log \frac{25 \times 47}{29}$

2. Express $\log x - 2 \log x + 3 \log (x + 1) - \log (x^2 - 1)$ as a single logarithm.
3. Write the following in the form of a single logarithm.
   (i) log 21 + log 5 (ii) log 25 – 2 log 3
   (iii) $2 \log x - 3 \log y$ (iv) log 5 + log 6 – log2
4. Calculate the following:
   (i) $\log_3 2 \times \log_2 81$ (ii) $\log_5 3 \times \log_3 25$
5. If log 2 = 0.3010, log 3 = 0.4771, log 5 = 0.6990, then find the values of the following
   (i) log 32 (ii) log 24 (iii) $\log \sqrt{3\frac{1}{3}}$
   (iv) $\log \frac{8}{3}$ (v) log 30

3.5 Application of Laws of Logarithm in Numerical Calculations

So far we have applied laws of logarithm to simple type of products, quotients, powers or roots of numbers. We now extend their application to more difficult examples to verify their effectiveness in simplification.

Example 1

Show that

$$7 \log \frac{16}{15} + 5 \log \frac{25}{24} + \log \frac{81}{80} = \log 2.$$

Solution

L.H.S. $= 7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80}$

= 7[log 16 – log 15] + 5[log 25 – log 24] + 3[log 81 – log 80]
= $7[\log 2^4 - \log (3 \times 5)] + 5[\log 5^2 - \log (2^3 \times 3)] + 3[\log 3^4 - \log (2^4 \times 5)]$
= 7[4 log 2 – log 3 – log 5] + 5[2 log 5 – 3 log 2 – log 3] + 3[4 log 3 – 4 log2 – log 5]
= (28 – 15 – 12) log 2 + (–7 – 5 + 12) log 3 + (–7 + 10 – 3)log 5
= log 2 + 0 + 0 = log 2 = R.H.S.

Example 2

Evaluate: $\sqrt[3]{\frac{0.07921 \times (18.99)^2}{(5.79)^4 \times 0.9474}}$

Solution

Let $y = \sqrt[3]{\frac{0.07921 \times (18.99)^2}{(5.79)^4 \times 0.9474}} = \left(\frac{0.07921 \times (18.99)^2}{(5.79)^4 \times 0.9474}\right)^{1/3}$

Then $\log y = \frac{1}{3} \log \left(\frac{0.07921 \times (18.99)^2}{(5.79)^4 \times 0.9474}\right)$

$= \frac{1}{3} [\log \{0.07921 \times (18.99)^2\} - \log \{(5.79)^4 \times 0.9474\}]$

$= \frac{1}{3} [\log 0.07921 + 2 \log 18.99 - 4 \log 5.79 - \log 0.9474]$

$$= \frac{1}{3}[\bar{2}.8988 + 2(1.2786) - 4(0.7627) - \bar{1}.9765]$$

$$= \frac{1}{3}[\ \bar{2}.8988 + 2.5572 - 3.0508 - \bar{1}.9765]$$

$$= \frac{1}{3}[1.4560 - 3.0273] = \frac{1}{3}(\ \bar{2}.4287)$$

$$= \frac{1}{3}(\ \bar{3} + 1.4287)$$

$$= \bar{1} + 0.4762 = \bar{1}.4762$$

or $y = \text{antilog}\ \bar{1}.4762 = 0.2993$

Example 3

Given $A = A_o e^{-kd}$. If $k = 2$, what should be the value of d to make $A = \frac{A_o}{2}$?

Solution

Given that $A = A_o e^{-kd}$. $\Rightarrow \frac{A}{A_o} = e^{-kd}$

Substituting $k = 2$, and $A = \frac{A_o}{2}$, we get $\frac{1}{2} = e^{-2d}$

Taking common log on both sides,

$\log_{10}1 - \log_{10}2 = -2d \log_{10}e$, where $e = 2.718$

$0 - 0.3010 = -2d\ (0.4343)$

$$d = \frac{0.3010}{2 \times 0.4343} = 0.3465$$

EXERCISE 3.4

1. Use log tables to find the value of

(i) 0.8176×13.64 (ii) $(789.5)^{1/8}$ (iii) $\frac{0.678 \times 9.01}{0.0234}$

(iv) $\sqrt[5]{2.709} \times \sqrt[7]{1.239}$ (v) $\frac{(1.23)\ (0.6975)}{(0.0075)\ (1278)}$ (vi) $\sqrt[3]{\frac{0.7214 \times 20.37}{60.8}}$

(vii) $\frac{83 \times \sqrt[3]{92}}{127 \times \sqrt[5]{246}}$ (viii) $\frac{(438)^3 \sqrt{0.056}}{(388)^4}$

2. A gas is expanding according to the law $pv^n = C$. Find C when $p = 80$, $v = 3.1$ and $n = \frac{5}{4}$.

3. The formula $p = 90\ (5)^{-q/10}$ applies to the demand of a product, where q is the number of units and p is the price of one unit. How many units will be demanded if the price is Rs 18.00?

4. If $A = \pi r^2$, find A, when $\pi = \frac{22}{7}$ and $r = 15$

5. If $V = \frac{1}{3}\pi r^2 h$, find V, when $\pi = \frac{22}{7}$, $r = 2.5$ and $h = 4.2$

REVIEW EXERCISE 3

1. Multiple Choice Questions. Choose the correct answer.

2. Complete the following:

(i) For common logarithm, the base is ....

(ii) The integral part of the common logarithm of a number is called the .....

(iii) The decimal part of the common logarithm of a number is called the ....

(iv) If $x = \log y$, then y is called the .... of x.

(v) If the charactcristic of the logarithm of a number is 2, that number will have .... zero(s) immediately after the decimal point.

(vi) If the characteristic of the logarithm of a number is 1, that number will have digits in its integral part.

3. Find the value of x in the following:

(i) $\log_3 x = 5$ (ii) $\log_4 256 = x$

(iii) $\log_{625} 5 = \frac{1}{4}x$ (iv) $\log_{64} x = \frac{-2}{3}$

4. Find the value of x in the following:
   (i) $\log x = 2.4543$ (ii) $\log x = 0.1821$
   (iii) $\log x = 0.0044$ (iv) $\log x = 1.6238$
5. If $\log 2 = 0.3010$, $\log 3 = 0.4771$ and $\log 5 = 0.6990$, then find the values of the following:

   (i) $\log 45$ (ii) $\log \frac{16}{15}$ (iii) $\log 0.048$

6. Simplify the following:

   (i) $\sqrt[3]{25.47}$ (ii) $\sqrt[5]{342.2}$ (iii) $\dfrac{(8.97)^3 \times (3.95)^2}{\sqrt[3]{15.37}}$

SUMMARY

- If $a^x = y$, then x is called the **logarithm** of y to the base a and is written as $x = \log_a y$, where $a > 0$, $a \neq 1$ and $y > 0$.
- If $x = \log_a y$, then $a^x = y$.
- If the base of the logarithm is taken as 10, it is known as common logarithm and if the base is taken as e($\approx$2.718) then it is known as natural or Naperian logarithm.
- The integral part of the common logarithm of a number is called the characteristic and the decimal part the mantissa.
- (i) For a number greater than 1, the characteristic of its logarithm
- is equal to the number of digits in the integral part of the number minus one.
- (ii) For a number less than 1, the characteristic of its logarithm is always negative and is equal to the number of zeros immediately after the decimal point of the number plus one.
- When a number is less than 1, the characteristic is always written as $\bar{3}, \bar{2}, \bar{1}$ (instead of –3, –2, –1) to avoid the mantissa becoming negative
- The logarithms of numbers having the same sequence of significant digits have the same mantissa.

- The number corresponding to a given logarithm is known as antilogarithm.
- $\log_e 10 = 2.3026$ and $\log_{10} e = 0.4343$
- Laws of logarithms.
  (i) $\log_a (mn) = \log_a m + \log_a n$
  (ii) $\log_a (\frac{m}{n}) = \log_a m - \log_a n$
  (iii) $\log_a (m^n) = n \log_a m$
  (iv) $\log_a n = \log_b n - \log_a b$

CHAPTER 4

ALGEBRAIC EXPRESSIONS AND ALGEBRAIB FORMULAS

Animation 4.1: Algebraic Expressions and Algebraic Formulas
Source & Credit: eLearn.punjab

Students Learning Outcomes

After studying this unit, the students will be able to:

- Know that a rational expression behaves like a rational number.
- Define a rational expression as the quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$ where $q(x)$ is not the zero polynomial.
- Examine whether a given algebraic expression is a
  - polynomial or not,
  - rational expression or not.
- Define $\frac{p(x)}{q(x)}$ as a rational expression in its lowest terms if $p(x)$ and $q(x)$ are polynomials with integral coefficients and having no common factor.
- Examine whether a given rational algebraic expression is in lowest from or not.
- Reduce a given rational expression to its lowest terms.
- Find the sum, difference and product of rational expressions.
- Divide a rational expression with another and express the result in it lowest terms.
- Find value of algebraic expression for some particular real number. Know the formulas
  $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$,
  $(a + b)^2 - (a - b)^2 = 4ab$
- Find the value of $a^2 + b^2$ and of ab when the values of $a + b$ and $a - b$ are known.
- Know the formulas
  $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.
- find the value of $a^2 + b^2 + c^2$ when the values of $a + b + c$ and $ab + bc + ca$ are given.
- find the value of $a + b + c$ when the values of $a^2 + b^2 + c^2$ and $ab + bc + ca$ are given.
- find the value of $ab + bc + ca$ when the values of $a^2 + b^2 + c^2$ and $a + b + c$ are given.

- know the formulas
  $(a + b)^3 = a^3 + 3ab(a + b) + b^3$,
  $(a - b)^3 = a^3 - 3ab(a - b) - b^3$,
- find the value of $a^3 \pm b^3$ when the values of $a \pm b$ and ab are given
- find the value of $x^3 \pm$ when the value of $x \pm$ is given.
- know the formulas
  $a^3 \pm b^3 = (a \pm b)(a^2 \pm ab + b^2)$.
  - find the product of $x+\frac{1}{x}$ and $x^2+\frac{1}{x^2}-1$.
  - find the product of $x-\frac{1}{x}$ and $x^2+\frac{1}{x^2}+1$.
  - find the continued product of
    $(x + y)(x - y)(x^2 + xy + y^2)(x^2 - xy + y^2)$.
- recognize the surds and their application.
- explain the surds of second order. Use basic operations on surds of second order to rationalize the denominators and evaluate it.
- explain rationalization (with precise meaning) of real numbers of the types $\frac{1}{a+b\sqrt{x}}$, $\frac{1}{\sqrt{x}+\sqrt{y}}$ and their combinations where x and y are natural numbers and a and b integers.

4.1 Algebraic Expressions

Algebra is a generalization of arithmetic. Recall that when operations of addition and subtraction are applied to algebraic terms, we obtain an algebraic expression. For instance, $5x^2 - 3x + \frac{2}{\sqrt{x}}$ and $3xy + \frac{3}{x}$ $(x \neq 0)$ are algebraic expressions.

Polynomials

A polynomial in the variable x is an algebraic expression of the form $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1x + a_0$, $a_n \neq 0$ ...... (i)

where n, the highest power of x, is a non-negative integer called the degree of the polynomial and each coefficient a_n, is a real number. The coefficient a_n of the highest power of x is called the *leading coefficient* of the polynomial. $2x^4y^3 + x^2y^2 + 8x$ is a polynomial in two variables x and y and has degree 7.

From the study of similar properties of integers and polynomials w.r.t. addition and multiplication, we may say that polynomials behave like integers.

Self Testing

Justify the following as polynomial or not a polynomial.

(i) $3x^2 + 8x + 5$ (ii) $x^3+\sqrt{2}x^2+5x-3$

(iii) $x^2+\sqrt{x}-4$ (iv) $\dfrac{3x^2+2x+8}{3x+4}$

4.1.1 Rational Expressions Behave like Rational Numbers

Let a and b be two integers, then $\frac{a}{b}$ is not necessarily an integer.

Therefore, number system is extended and $\frac{a}{b}$ is defined as a rational number where $a, b \in Z$ and $b \neq 0$.

Similarly, if $p(x)$ and $q(x)$ are two polynomials, the $\dfrac{p(x)}{q(x)}$ is not necessarily a polynomial, where $q(x) \neq 0$. Therefore, similar to the idea of rational numbers, concept of rational expressions is developed.

4.1.2 Rational Expression

The quotient $\dfrac{p(x)}{q(x)}$ of two polynomials, $p(x)$ and $q(x)$, where $q(x)$ is a non-zero polynomial, is called a rational expression.

For example, $\dfrac{2x+1}{3x+8}$, $3x + 8 \neq 0$ is a rational expression.

In the rational expression $\dfrac{p(x)}{q(x)}$, $p(x)$ is called the numerator and $q(x)$ is known as the denominator of the rational expression $\dfrac{p(x)}{q(x)}$. The rational expression $\dfrac{p(x)}{q(x)}$ need not be a polynomial.

Note:

Every polynomial $p(x)$ can be regarded as a rational expression, since we can write $p(x)$ as $\dfrac{p(x)}{1}$.Thus, every polynomial is a rational expression, but every rational expression need not be a polynomial.

Self Testing

Identify the following as a rational expression or not a rational expression.

(i) $\dfrac{2x+6}{3x-4}$ (ii) $\dfrac{3x+8}{x^2+x+2}$ (iii) $\dfrac{x^2+4x+5}{x^2+3\sqrt{x}+4}$ (iv) $\dfrac{\sqrt{x}}{3x^2+1}$

4.1.3 Properties of Rational Expressions

The method for operations with rational expressions is similar to operations with rational numbers.

Let $p(x), q(x), r(x), s(x)$ be any polynomials such that all values of the variable that make a rational expression undefined are excluded from the domain. Then following properties of rational expressions hold under the supposition that they all are defined (i.e., denominator (s) $\neq 0$).

(i) $\dfrac{p(x)}{q(x)}=\dfrac{r(x)}{s(x)}$ if and only if $p(x)\,s(x) = q(x)\,r(x)$ (Equality)

(ii) $\dfrac{p(x)k}{q(x)k}=\dfrac{p(x)}{q(x)}$ (Cancellation)

(iii) $\dfrac{p(x)}{q(x)}+\dfrac{r(x)}{r(x)}=\dfrac{p(x)\,s(x)+q(x)\,r(x)}{q(x)\,s(x)}$ (Addition)

(iv) $\frac{p(x)}{q(x)} - \frac{r(x)}{s(x)} = \frac{p(x)\,s(x) - q(x)\,r(x)}{q(x)\,s(x)}$ (Subtraction)

(v) $\frac{p(x)}{q(x)} \cdot \frac{r(x)}{s(x)} = \frac{p(x)\,r(x)}{q(x)\,s(x)}$ (Multiplication)

(vi) $\frac{p(x)}{q(x)} \div \frac{r(x)}{s(x)} = \frac{p(x)}{q(x)} \cdot \frac{s(x)}{r(x)} = \frac{p(x)\,s(x)}{q(x)\,r(x)}$ (Division)

(vii) **Additive inverse** of $\frac{p(x)}{q(x)}$ is $-\frac{p(x)}{q(x)}$

(viii) **Multiplicative inverse** or reciprocal of $\frac{p(x)}{q(x)}$ is $\frac{q(x)}{p(x)}$, $p(x) \neq 0$, $q(x) \neq 0$.

4.1.4 Rational Expression in its Lowest form

The rational expression $\frac{p(x)}{q(x)}$ is said to be in its lowest form, if *p(x)* and *q(x)* are polynomials with integral coefficients and have no common factor.

For example, $\frac{x+1}{x^2+1}$ is in its lowest form.

4.1.5 To examine whether a rational expression is in lowest form or not

To examine the rational expression $\frac{p(x)}{q(x)}$, find H.C.F of $p(x)$ and $q(x)$. If H.C.F is 1, then the rational expression is in lowest form.

For example, $\frac{x-1}{x^2+1}$ is in its lowest form as H.C.F. of $x - 1$ and $x^2 + 1$ is 1.

4.1.6 Working Rule to reduce a rational expression to its lowest terms

Let the given rational expression be $\frac{p(x)}{q(x)}$

Step I Factorize each of the two polynomials $p(x)$ and $q(x)$.

Step II Find H.C. F. of $p(x)$ and $q(x)$.

Step III Divide the numerator $p(x)$ and the denominator $q(x)$ by the H.C. F. of $p(x)$ and $q(x)$. The rational expression so obtained, is in its lowest terms.

In other words, an algebraic fraction can be reduced to its lowest form by first factorizing both the polynomials in the numerator and the denominator and then cancelling the common factors between them.

Example

Reduce the following algebraic fractions to their lowest form.

(i) $\frac{lx + mx - ly - my}{3x^2 - 3y^2}$ (ii) $\frac{3x^2 + 18x + 27}{5x^2 - 45}$

Solution

(i) $\frac{lx + mx - ly - my}{3x^2 - 3y^2} = \frac{x(l+m) - y(l+m)}{3(x^2 - y^2)}$

$= \frac{(l+m)(x-y)}{3(x+y)(x-y)}$ ...... (factorizing)

$= \frac{l+m}{3(x+y)}$ ...... (cancelling common factors)

which is in the lowest form

(ii) $\frac{3x^2 + 18x + 27}{5x^2 - 45} = \frac{3(x^2 + 6x + 9)}{5(x^2 - 9)}$ ...... *(monomial factors)*

$= \frac{3(x+3)(x+3)}{5(x+3)(x-3)}$ ...... *(factorizing)*

$= \frac{3(x+3)}{5(x-3)}$ ...... *(cancelling common factors)*

which is in the lowest form.

4.1.7 Sum, Difference and Product of Rational Expressions

For finding sum and difference of algebraic expressions

containing rational expressions, we take the L.C.M. of the denominators and simplify as explained in the following examples by using properties stated in 4.1.3.

Example 1

Simplify (i) $\frac{1}{x-y}-\frac{1}{x+y}+\frac{2x}{x^2-y^2}$ (ii) $\frac{2x^2}{x^4-16}-\frac{x}{x^2-4}+\frac{1}{x+2}$

Solution

(i) $\frac{1}{x-y}-\frac{1}{x+y}+\frac{2x}{x^2-y^2}=\frac{1}{x-y}-\frac{1}{x+y}+\frac{2x}{(x+y)(x-y)}$

$=\frac{x+y-(x-y)+2x}{(x+y)(x-y)}$ (L.C.M. of denominators)

$=\frac{x+y-x+y+2x}{(x+y)(x-y)}$

$=\frac{2x+2y}{(x+y)(x-y)}$ (simplifying)

$=\frac{2(x+y)}{(x+y)(x-y)}=\frac{2}{x-y}$ (cancelling common factors)

(ii) $\frac{2x^2}{x^4-16}-\frac{x}{x^2-4}+\frac{1}{x+2}$

$=\frac{2x^2}{(x^2+4)(x^2-4)}-\frac{x}{x^2-4}+\frac{1}{x+2}$ (difference of two squares)

$=\frac{2x^2}{(x^2+4)(x+2)(x-2)}-\frac{x}{(x+2)(x-2)}+\frac{1}{x+2}$

$=\frac{2x^2-x(x^2+4)+(x^2+4)(x-2)}{(x^2+4)(x+2)(x-2)}=\frac{2x^2-x^3-4x+x^3+4x-2x^2-8}{(x^2+4)(x+2)(x-2)}$

$=\frac{-8}{(x^2+4)(x+2)(x-2)}$ (on simplification)

Example 2

Find the product $\frac{x+2}{2x-3y}\cdot\frac{4x^2-9y^2}{xy+2y}$ (in simplified form)

Solution

$\frac{x+2}{2x-3y}\cdot\frac{4x^2-9y^2}{xy+2y}=\frac{(x+2)\,[(2x)^2-(3y)^2]}{(2x-3y)\,(x+2)y}$ (monomial factors)

$=\frac{(x+2)\,(2x+3y)\,(2x-3y)}{y(x+2)\,(2x-3y)}$ (factorizing)

$=\frac{2x+3y}{y}$ (reduced to the lowest forms)

4.1.8 Dividing a Rational Expression with another Rational Expression

In order to divide one rational expression with another, we first invert for changing division to multiplication and simplify the resulting product to the lowest terms.

Example

Simplify $\frac{7xy}{x^2-4x+4}\div\frac{14y}{x^2-4}$

Solution

$\frac{7xy}{x^2-4x+4}\div\frac{14y}{x^2-4}$

$=\frac{7xy}{x^2-4x+4}\cdot\frac{x^2-4}{14y}$ ...(changing division into multiplication)

$=\frac{7xy}{(x-2)\,(x-2)}\cdot\frac{(x+2)\,(x-2)}{14y}$ ...(factorizing)

$=\frac{x(x+2)}{2(x-2)}$ ...(reduced to lowest forms)

4.1.9 Evaluation of Algebraic Expression for some particular Real Number Definition

If specific numbers are substituted for the variables in an algebraic expression, the resulting number is called the value of the expression.

Example

Evaluate $\dfrac{3x^2\sqrt{y}+6}{5(x+y)}$ if $x = -4$ and $y = 9$

Solution

We have, by putting $x = -4$ and $y = 9$,

$$=\frac{3x^2\sqrt{y}+6}{5(x+y)}=\frac{3(-4)^2\sqrt{9}+6}{5(-4+9)}=\frac{3(16)(3)+6}{5(5)}=\frac{150}{25}=6$$

EXERCISE 4.1

1. Identify whether the following algebraic expressions are polynomials (Yes or No).

(i) $3x^2+\dfrac{1}{x}-5$ (ii) $3x^3-4x^2-x\sqrt{x}+3$

(iii) $x^2-3x+\sqrt{2}$ (iv) $\dfrac{3x}{2x-1}+8$

2. State whether each of the following expressions is a rational expression or not.

(i) $\dfrac{3\sqrt{x}}{3\sqrt{x}+5}$ (ii) $\dfrac{x^3-2x^2+\sqrt{3}}{2+3x-x^2}$

(iii) $\dfrac{x^2+6x+9}{x^2-9}$ (iv) $\dfrac{2\sqrt{x}+3}{2\sqrt{x}-3}$

3. Reduce the following rational expressions to the lowest form.

(i) $\dfrac{120x^2y^3z^5}{30x^3yz^2}$ (ii) $\dfrac{8a(x+1)}{2(x^2-1)}$

(iii) $\dfrac{(x+y)^2-4xy}{(x-y)^2}$ (iv) $\dfrac{(x^3-y^3)(x^2-2xy+y^2)}{(x-y)(x^2+xy+y^2)}$

(v) $\dfrac{(x+2)(x^2-1)}{(x+1)(x^2-4)}$ (vi) $\dfrac{x^2-4x+4}{2x^2-8}$

(vii) $\dfrac{64x^5-64x}{(8x^2+8)(2x+2)}$ (viii) $\dfrac{9x^2-(x^2-4)^2}{4+3x-x^2}$

4. Evaluate (a) $\dfrac{x^3y-2z}{xz}$ for

(i) $x = 3, y = -1, z = -2$ (ii) $x = -1, y = -9, z = 4$

(b) for $x = 4, y = -2, z = -1$

5. Perform the indicated operation and simplify.

(i) $\dfrac{15}{2x-3y}-\dfrac{4}{3y-2x}$ (ii) $\dfrac{1+2x}{1-2x}-\dfrac{1-2x}{1+2x}$

(iii) $\dfrac{x^2-25}{x^2-36}-\dfrac{x+5}{x+6}$ (iv) $\dfrac{x}{x-y}-\dfrac{y}{x+y}-\dfrac{2xy}{x^2-y^2}$

(v) $\dfrac{x-2}{x^2+6x+9}-\dfrac{x+2}{2x^2-18}$ (vi) $\dfrac{1}{x-1}-\dfrac{1}{x+1}-\dfrac{2}{x^2+1}-\dfrac{4}{x^4-1}$

6. Perform the indicated operation and simplify.

(i) $(x^2-49).\dfrac{5x+2}{x+7}$ (ii) $\dfrac{4x-12}{x^2-9}\div\dfrac{18-2x^2}{x^2+6x+9}$

(iii) $\dfrac{x^6-y^6}{x^2-y^2}\div(x^4+x^2y^2+y^4)$ (iv) $\dfrac{(x^3-y^3)(x^2-2xy+y^2)}{(x-y)(x^2+xy+y^2)}$

(v) $\dfrac{x^2-1}{x^2+2x+1}.\dfrac{x+5}{1-x}$

4.2 Algebraic Formulae

4.2.1 Using the formulas

(i) $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$ and $(a + b)^2 - (a - b)^2 = 4ab$

The process of finding the values of $a^2 + b^2$ and abis explained in the following examples.

$\Rightarrow$ $(7)^2 - (3)^2 = 4ab$ ....(substituting given values)

$\Rightarrow$ $49 - 9 = 4ab$

Example

If $a + b = 7$ and $a - b = 3$, then find the value of **(a)** $a^2 + b^2$ **(b)** ab

Solution

We are given that $a + b = 7$ and $a - b = 3$

(a) To find the value of $(a^2 + b^2)$, we use the formula

$$(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$$

Substituting the values $a + b = 7$ and $a - b = 3$, we get

$$(7)^2 + (3)^2 = 2(a^2 + b^2)$$
$$\Rightarrow \quad 49 + 9 = 2(a^2 + b^2)$$
$$\Rightarrow \quad 58 = 2(a^2 + b^2) \quad \text{....(simplifying)}$$
$$\Rightarrow \quad 29 = a^2 + b^2 \quad \text{....(dividing by 2)}$$

(b) To find the value of ab, we make use of the formula

$$(a + b)^2 - (a - b)^2 = 4ab$$
$$(7)^2 - (3)^2 = 4ab$$
$$\Rightarrow \quad 49 - 9 = 4ab$$
$$\Rightarrow \quad 40 = 4ab \quad \text{....(simplifying)}$$
$$\Rightarrow \quad 10 = ab \quad \text{....(dividing by 4)}$$

Hence $a^2 + b^2 = 29$ and $ab = 10$.

(ii) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

This formula, square of a trinomial, involves three expressions, namely; $(a + b + c)$, $(a^2 + b^2 + c^2)$ and $2(ab + bc + ca)$. If the values of two of them are known, the value of the third expression can be calculated. The method is explained in the following examples.

Example 1

If $a^2 + b^2 + c^2 = 43$ and $ab + bc + ca = 3$, then find the value of $a + b + c$.

Solution

We know that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$
$$\Rightarrow \quad (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

$$\Rightarrow \quad (a + b + c)^2 = 43 + 2 \times 3 \quad (\text{Putting } a^2 + b^2 + c^2 = 43 \text{ and } ab + bc + ca = 3)$$
$$\Rightarrow \quad (a + b + c)^2 = 49$$
$$\Rightarrow \quad a + b + c = \pm\sqrt{49}$$

Hence $a + b + c = \pm 7$

Example 2

If $a + b + c = 6$ and $a^2 + b^2 + c^2 = 24$, then find the value of $ab + bc + ca$.

Solution

We have

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$
$$\therefore \quad (6)^2 = 24 + 2(ab + bc + ca)$$
$$\Rightarrow \quad 36 = 24 + 2(ab + bc + ca)$$
$$\Rightarrow \quad 12 = 2(ab + bc + ca)$$

Hence $ab + bc + ca = 6$

Example 3

If $a + b + c = 7$ and $ab + bc + ca = 9$, then find the value of $a^2 + b^2 + c^2$.

Solution

We know that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$
$$\Rightarrow \quad (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$
$$\Rightarrow \quad (7)^2 = a^2 + b^2 + c^2 + 2(9)$$
$$\Rightarrow \quad 49 = a^2 + b^2 + c^2 + 18$$
$$\Rightarrow \quad 31 = a^2 + b^2 + c^2$$

Hence $a^2 + b^2 + c^2 = 31$

(iii) **$(a + b)^3 = a^3 + 3ab(a + b) + b^3$**

$(a - b)^3 = a^3 - 3ab(a - b) - b^3$

Example 1

If $2x - 3y = 10$ and $xy = 2$, then find the value of $8x^3 - 27y^3$

Solution

We are given that $2x - 3y = 10$

$\Rightarrow \quad (2x-3y)^3 = (10)^3$

$\Rightarrow \quad 8x^3 - 27y^3 - 3 \times 2x \times 3y(2x-3y) = 1000$

$\Rightarrow \quad 8x^3 - 27y^3 - 18xy(2x-3y) = 1000$

$\Rightarrow \quad 8x^3 - 27y^3 - 18 \times 2 \times 10 = 1000$

$\Rightarrow \quad 8x^3 - 27y^3 - 360 = 1000$

Hence $\quad 8x^3 - 27y^3 = 1360$

Example 2

If $x + \frac{1}{x} = 8$, then find the value of $x^3 + \frac{1}{x^3}$

Solution

We have been given $x + \frac{1}{x} = 8$

$\Rightarrow \quad \left(x + \frac{1}{x}\right)^3 = (8)^3$

$\Rightarrow \quad x^3 + \frac{1}{x^3} + 3 \times x \times \frac{1}{x}\left(x + \frac{1}{x}\right) = 512$

$\Rightarrow \quad x^3 + \frac{1}{x^3} + 3 \times \left(x + \frac{1}{x}\right) = 512$

$\Rightarrow \quad x^3 + \frac{1}{x^3} + 3 \times 8 = 512$

$\Rightarrow \quad x^3 + \frac{1}{x^3} + 24 = 512$

Hence $\quad x^3 + \frac{1}{x^3} = 488$

Example 3

If $x - \frac{1}{x} = 4$, then find

Solution

We have

$\Rightarrow \quad \left(x - \frac{1}{x}\right)^3 = 64$

$\Rightarrow \quad x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right) = 64$

$\Rightarrow \quad x^3 - \frac{1}{x^3} - 3(4) = 64$

$\Rightarrow \quad x^3 - \frac{1}{x^3} - 12 = 64$

$\Rightarrow \quad x^3 - \frac{1}{x^3} = 64 + 12$

$\Rightarrow \quad x^3 - \frac{1}{x^3} = 76$

(iv) $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

The procedure for finding the products of $\left(x \pm \frac{1}{x}\right)$ and $x^2 + \frac{1}{x^2} \mp 1$ is also explained in following examples.

Example 1

Factorize $64x^3 + 343y^3$

Solution

We have

$64x^3 + 343y^3 = (4x)^3 + (7y)^3$

$= (4x + 7y)[(4x)^2 - (4x)(7y) + (7y)^2]$

$= (4x + 7y)(16x^2 - 28xy + 49y^2)$

Example 2

Factorize $125x^3 - 1331y^3$

Solution

We have

$$125x^3 - 1331y^3 = (5x)^3 - (11y)^3$$
$$= (5x - 11y)\,[(5x)^2 + (5x)(11y) + (11y)^2]$$
$$= (5x - 11y)\,(25x^2 + 55xy + 121y^2)$$

Example 3

Find the product $\left(\frac{2}{3}x + \frac{3}{2x}\right)\left(\frac{4}{9}x^2 - 1 + \frac{9}{4x^2}\right)$

Solution

$$\left(\frac{2}{3}x + \frac{3}{2x}\right)\left(\frac{4}{9}x^2 - 1 + \frac{9}{4x^2}\right)$$
$$= \left(\frac{2}{3}x + \frac{3}{2x}\right)\left[\left(\frac{2}{3}x\right)^2 - \left(\frac{2}{3}x\right)\left(\frac{3}{2x}\right) + \left(\frac{3}{2x}\right)^2\right]$$
$$= \left(\frac{2}{3}x\right)^3 + \left(\frac{3}{2x}\right)^3$$
$$= \frac{8}{27}x^3 + \frac{27}{8x^3}$$

Example 4

Find the product $\left(\frac{4}{5}x - \frac{5}{4x}\right)\left(\frac{16}{25}x^2 + \frac{25}{16x^2} + 1\right)$

Solution

$$\left(\frac{4}{5}x - \frac{5}{4x}\right)\left(\frac{16}{25}x^2 + \frac{25}{16x^2} + 1\right)$$
$$= \left(\frac{4}{5}x - \frac{5}{4x}\right)\left(\frac{16x^2}{25} + 1 + \frac{25}{16x^2}\right) \quad \text{(rearranging)}$$
$$= \left(\frac{4}{5}x - \frac{5}{4x}\right)\left[\left(\frac{4}{5}x\right)^2 + \left(\frac{4}{5}x\right)\left(\frac{5}{4x}\right) + \left(\frac{5}{4x}\right)^2\right]$$
$$= \left(\frac{4}{5}x\right)^3 - \left(\frac{5}{4x}\right)^3 = \frac{64}{125}x^3 - \frac{125}{64x^3}$$

Example 5

Find the continued product of $(x + y)(x - y)(x^2 + xy + y^2)(x^2 - xy + y^2)$

Solution

$$(x + y)(x - y)(x^2 + xy + y^2)(x^2 - xy + y^2)$$
$$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2) \quad \text{(rearranging)}$$
$$= (x^3 + y^3)(x^3 - y^3) = (x^3)^2 - (y^3)^2 = x^6 - y^6$$

EXERCISE 4.2

1. (i) If $a + b = 10$ and $a - b = 6$, then find the value of $(a^2 + b^2)$
   (ii) If $a + b = 5$, $a - b = \sqrt{17}$, then find the value of ab.
2. If $a^2 + b^2 + c^2 = 45$ and $a + b + c = -1$, then find the value of $ab + bc + ca$.
3. If $m + n + p = 10$ and $mn + np + mp = 27$, then find the value of $m^2 + n^2 + p^2$.
4. If $x^2 + y^2 + z^2 = 78$ and $xy + yz + zx = 59$, then find the value of $x + y + z$.
5. If $x + y + z = 12$ and $x^2 + y^2 + z^2 = 64$, then find the value of $xy + yz + zx$.
6. If $x + y = 7$ and $xy = 12$, then find the value of $x^3 + y^3$.
7. If $3x + 4y = 11$ and $xy = 12$, then find the value of $27x^3 + 64y^3$.
8. If $x - y = 4$ and $xy = 21$, then find the value of $x^3 - y^3$.
9. If $5x - 6y = 13$ and $xy = 6$, then find the value of $125x^3 - 216y^3$.
10. If $x + \frac{1}{x} = 3$, then find the value of $x^3 + \frac{1}{x^3}$
11. If $x - \frac{1}{x} = 7$, then find the value of $x^3 - \frac{1}{x^3}$
12. If $\left(3x + \frac{1}{3x}\right) = 5$, then find the value of $\left(27x^3 + \frac{1}{27x^3}\right)$
13. If $\left(5x - \frac{1}{5x}\right) = 6$, then find the value of $\left(125x^3 - \frac{1}{125x^3}\right)$
14. Factorize (i) $x^3 - y^3 - x + y$ (ii) $8x^3 - \frac{1}{27y^3}$

15. Find the products, using formulas.
    (i) $(x^2 + y^2)(x^4 - x^2y^2 + y^4)$ (ii) $(x^3 - y^3)(x^6 + x^3y^3 + y^6)$
    (iii) $(x - y)(x + y)(x^2 + y^2)(x^2 + xy + y^2)(x^2 - xy + y^2)(x^4 - x^2y^2 + y^4)$
    (iv) $(2x^2 - 1)(2x^2 + 1)(4x^4 + 2x^2 + 1)(4x^4 - 2x^2 + 1)$

4.3 Surds and their Application

4.3.1 Definition

An irrational radical with rational radicand is called a surd.
Hence the radical is a surd if
(i) a is rational, (ii) the result is irrational.
e.g., $\sqrt{3}, \sqrt{2/5}, \sqrt[3]{7}, \sqrt[4]{10}$ are surds.
But $\sqrt{\pi}$ and $\sqrt{2+\sqrt{17}}$ are not surds because π and $2+\sqrt{17}$ are not rational.

> Note that for the surd $\sqrt[n]{a}$, n is called surd index or the order of the surd and the rational number 'a' is called the radicand. $\sqrt[3]{7}$ is third order surd.

Every surd is an irrational number but every irrational number is not a surd. e.g., the surd $\sqrt[3]{5}$ is an irrational but the irrational number $\sqrt{\pi}$ is not a surd.

4.3.2 Operations on surds

(a) Addition and Subtraction of Surds

Similar surds (i.e., surds having same irrational factors) can be added or subtracted into a single term is explained in the following examples.

Example

Simplify by combining similar terms.

(i) $4\sqrt{3} - 3\sqrt{27} + 2\sqrt{75}$. (ii) $\sqrt[3]{128} - \sqrt[3]{250} + \sqrt[3]{432}$

Solution

(i) $4\sqrt{3} - 3\sqrt{27} + 2\sqrt{75}$
$= 4\sqrt{3} - 3\sqrt{9\times 3} + 2\sqrt{25\times 3} = 4\sqrt{3} - 3\sqrt{9}\sqrt{3} + 2\sqrt{25}\times\sqrt{3}$
$= 4\sqrt{3} - 9\sqrt{3} + 10\sqrt{3} = (4 - 9 + 10)\sqrt{3} = 5\sqrt{3}$

(ii) $\sqrt[3]{128} - \sqrt[3]{250} + \sqrt[3]{432}$
$= \sqrt[3]{64\times 2} - \sqrt[3]{125\times 2} + \sqrt[3]{216\times 2}$
$= \sqrt[3]{(4)^3\times 2} - \sqrt[3]{(5)^3\times 2} + \sqrt[3]{(6)^3\times 2}$
$= \sqrt[3]{(4)^3}\sqrt[3]{2} - \sqrt[3]{(5)^3}\sqrt[3]{2} + \sqrt[3]{(6)^3}\sqrt[3]{2}$
$= 4\sqrt[3]{2} - 5\sqrt[3]{2} + 6\sqrt[3]{2} = (4 - 5 + 6)\sqrt[3]{2} = 5\sqrt[3]{2}$

(b) Multiplication and Division of Surds

We can multiply and divide surds of the same order by making use of the following laws of surds

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

and the result obtained will be a surd of the same order.
If surds to be multiplied or divided are not of the same order, they must be reduced to the surds of the same order.

Example

Simplify and express the answer in the simplest form.

(i) $\sqrt{14}\sqrt{35}$ (ii) $\dfrac{\sqrt[6]{12}}{\sqrt{3}\sqrt[3]{2}}$

Solution

(i) $\sqrt{14}\sqrt{35} = \sqrt{14\times 35} = \sqrt{7\times 2\times 7\times 5} = \sqrt{(7)^2\times 2\times 5}$
$= \sqrt{(7)^2\times 10} = \sqrt{(7)^2}\times\sqrt{10} = 7\sqrt{10}$

(ii) We have $\dfrac{\sqrt[6]{12}}{\sqrt{3}\sqrt[3]{2}}$.

For $\sqrt{3}\sqrt[3]{2}$ the L.C.M. of orders 2 and 3 is 6.

Thus $\sqrt{3} = (3)^{1/2} = (3)^{3/6} = \sqrt[6]{3^3} = \sqrt[6]{27}$

and $\sqrt[3]{2} = (2)^{1/3} = (2)^{2/6} = \sqrt[6]{(2)^2} = \sqrt[6]{4}$

Hence $\dfrac{\sqrt[6]{12}}{\sqrt{3}\sqrt[3]{2}} = \dfrac{\sqrt[6]{12}}{\sqrt[6]{27}\sqrt[6]{4}} = \dfrac{\sqrt[6]{12}}{\sqrt[6]{108}} = \sqrt[6]{\dfrac{12}{108}} = \sqrt[6]{\dfrac{1}{9}}$

Its simplest form is

$$\sqrt[6]{\left(\frac{1}{3}\right)^2} = \left(\frac{1}{3}\right)^{2/6} = \left(\frac{1}{3}\right)^{1/3} = \sqrt[3]{\frac{1}{3}}$$

EXERCISE 4.3

1. Express each of the following surd in the simplest form.
   (i) $\sqrt{180}$ (ii) $3\sqrt{162}$
   (iii) $\frac{3}{4}\sqrt[3]{128}$ (iv) $\sqrt[5]{96x^6y^7z^8}$
2. Simplify
   (i) $\dfrac{\sqrt{18}}{\sqrt{3}\sqrt{2}}$ (ii) $\dfrac{\sqrt{21}\sqrt{9}}{\sqrt{63}}$
   (iii) $\sqrt[5]{243x^5y^{10}z^{15}}$ (iv) $\frac{4}{5}\sqrt[3]{125}$
   (v) $\sqrt{21} \times \sqrt{7} \times \sqrt{3}$
3. Simplify by combining similar terms.
   (i) $\sqrt{45} - 3\sqrt{20} + 4\sqrt{5}$ (ii) $4\sqrt{12} + 5\sqrt{27} - 3\sqrt{75} + \sqrt{300}$
   (iii) $\sqrt{3}\,(2\sqrt{3} + 3\sqrt{3})$ (iv) $2(6\sqrt{5} - 3\sqrt{5})$
4. Simplify
   (i) $(3+\sqrt{3})(3-\sqrt{3})$ (ii) $(\sqrt{5}+\sqrt{3})^2$
   (iii) $(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$ (iv) $\left(\sqrt{2}+\frac{1}{\sqrt{3}}\right)\left(\sqrt{2}-\frac{1}{\sqrt{3}}\right)$
   (v) $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})(x+y)(x^2+y^2)$

4.4 Rationalization of Surds

(a) Definitions

(i) A surd which contains a single term is called a monomial surd. e.g., $\sqrt{2}, \sqrt{3}$ etc.

(ii) A surd which contains sum of two monomial surds or sum of a monomial surd and a rational number is called a binomial surd. e.g., $\sqrt{3}+\sqrt{7}$ or $\sqrt{2}+5$ or $\sqrt{11}-8$ etc.
We can extend this to the definition of a trinomial surd.

(iii) If the product of two surds is a rational number, then each surd is called the rationalizing factor of the other.

(iv) The process of multiplying a given surd by its rationalizing factor to get a rational number as product is called rationalization of the given surd.

(v) Two binomial surds of second order differing only in sign connecting their terms are called conjugate surds. Thus $(\sqrt{a}+\sqrt{b})$ and $(\sqrt{a}-\sqrt{b})$ are conjugate surds of each other.
The conjugate of $x+\sqrt{y}$ is $x-\sqrt{y}$.
The product of the conjugate surds $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$,
$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a-b$,
is a rational quantity independent of any radical.
Similarly, the product of $a+b\sqrt{m}$ and its conjugate $a-b\sqrt{m}$ has no radical. For example,
$(3+\sqrt{5})(3-\sqrt{5}) = (3)^2 - (\sqrt{5})^2 = 9-5 = 4$, which is a rational number.

(b) Rationalizing a Denominator

Keeping the above discussion in mind, we observe that, in order to rationalize a denominator of the form $a+b\sqrt{x}$ (or $a-b\sqrt{x}$), we multiply both numerator and denominator by the conjugate factor $a-b\sqrt{x}$ (or $a+b\sqrt{x}$). By doing this we eliminate the radical and thus obtain a denominator free of any surd.

(c) Rationalizing Real Numbers of the Types $\frac{1}{a+b\sqrt{x}}$, $\frac{1}{\sqrt{x}+\sqrt{y}}$

For the expressions $\frac{1}{a+b\sqrt{x}}$, $\frac{1}{\sqrt{x}+\sqrt{y}}$ and their combinations, where *x, y* are natural numbers and *a, b* are integers, rationalization is explained with the help of following examples.

Example 1

Rationalize the denominator $\frac{58}{7-2\sqrt{5}}$

Solution

To rationalize the denominator, we multiply both the numerator and denominator by the conjugate $(7+2\sqrt{5})$ of $(7-2\sqrt{5})$, i.e.,

$$\frac{58}{7-2\sqrt{5}}=\frac{58}{7-2\sqrt{5}}\times\frac{7+2\sqrt{5}}{7+2\sqrt{5}}=\frac{58(7+2\sqrt{5})}{(7)^2-(2\sqrt{5})^2}$$

$$=\frac{58(7+2\sqrt{5})}{49-20};\ \text{(radical is eliminated in the denominator)}$$

$$=\frac{58(7+2\sqrt{5})}{29}=2(7+2\sqrt{5})$$

Example 2

Rationalize the denominator $\frac{2}{\sqrt{5}+\sqrt{2}}$

Solution

Multiply both the numerator and denominator by the conjugate $\sqrt{5}-\sqrt{2}$ of $\sqrt{5}+\sqrt{2}$, to get

$$\frac{2}{\sqrt{5}+\sqrt{2}}=\frac{2}{\sqrt{5}+\sqrt{2}}\times\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}=\frac{2\sqrt{5}-\sqrt{2}}{5-2}$$

$$=\frac{2(\sqrt{5}-\sqrt{2})}{3}=\frac{2(\sqrt{5}-\sqrt{2})}{3}$$

Example 3

Simplify $\frac{6}{2\sqrt{3}-\sqrt{6}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}-\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$

Solution

First we shall rationalize the denominators and then simplify. We have

$$\frac{6}{2\sqrt{3}-\sqrt{6}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}-\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$$

$$=\frac{6}{2\sqrt{3}-\sqrt{6}}\times\frac{2\sqrt{3}+\sqrt{6}}{2\sqrt{3}+\sqrt{6}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}-\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}\times\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}$$

$$=\frac{6(2\sqrt{3}+\sqrt{6})}{(2\sqrt{3})^2-(\sqrt{6})^2}+\frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{(\sqrt{3})^2-(\sqrt{2})^2}-\frac{4\sqrt{3}(\sqrt{6}+\sqrt{2})}{(\sqrt{6})^2-(\sqrt{2})^2}$$

$$=\frac{6(2\sqrt{3}+\sqrt{6})}{12-6}+\frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{3-2}-\frac{4\sqrt{3}(\sqrt{6}+\sqrt{2})}{6-2}$$

$$=\frac{12\sqrt{3}+6\sqrt{6}}{6}+\frac{\sqrt{6}\sqrt{3}-\sqrt{6}\sqrt{2}}{1}-\frac{4\sqrt{3}\sqrt{6}+4\sqrt{3}\sqrt{2}}{4}$$

$$=2\sqrt{3}+\sqrt{6}+3\sqrt{2}-2\sqrt{3}-3\sqrt{2}-\sqrt{6}=0$$

Example 4

Find rational numbers x and y such that $\frac{4+3\sqrt{5}}{4-3\sqrt{5}}=x+y\sqrt{5}$

Solution

We have

$$\frac{4+3\sqrt{5}}{4-3\sqrt{5}}=\frac{4+3\sqrt{5}}{4-3\sqrt{5}}\times\frac{4+3\sqrt{5}}{4+3\sqrt{5}}=\frac{(4+3\sqrt{5})^2}{(4)^2-(3\sqrt{5})^2}$$

$$=\frac{16+24\sqrt{5}+45}{16-45}=\frac{61+24\sqrt{5}}{-29}$$

$$\Rightarrow\quad \frac{-61}{29}-\frac{24}{29}\sqrt{5}=x+y\sqrt{5}\qquad\text{(given)}$$

Hence, an comparing the two sides, we get

$$x = -\frac{61}{29},\ y = -\frac{24}{29}$$

Example 5

If $x = 3+\sqrt{8}$, then evaluate

(i) $x+\frac{1}{x}$ and (ii) $x^2+\frac{1}{x^2}$

Solution

Since $x = 3+\sqrt{8}$, therefore,

$$\frac{1}{x} = \frac{1}{3+\sqrt{8}} = \frac{1}{3+\sqrt{8}} \times \frac{3-\sqrt{8}}{3-\sqrt{8}} = \frac{3-\sqrt{8}}{(3)^2-(\sqrt{8})^2}$$

$$= \frac{3-\sqrt{8}}{9-8} = 3-\sqrt{8}$$

(i) $x+\frac{1}{x} = 3+\sqrt{8}+3-\sqrt{8} = 6$

(ii) $\left(x+\frac{1}{x}\right)^2 = 36$

or $x^2+2x\times\frac{1}{x}+\frac{1}{x^2} = 36$

or $x^2+\frac{1}{x^2} = 34$

EXERCISE 4.4

1. Rationalize the denominator of the following.

(i) $\frac{3}{4\sqrt{3}}$ (ii) $\frac{14}{\sqrt{98}}$ (iii) $\frac{6}{\sqrt{8}\sqrt{27}}$ (iv) $\frac{1}{3+2\sqrt{5}}$

(v) $\frac{15}{\sqrt{31}-4}$ (vi) $\frac{2}{\sqrt{5}-\sqrt{3}}$ (vii) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ (viii) $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$

2. Find the conjugate of $x+\sqrt{y}$.

(i) $3+\sqrt{7}$ (ii) $4-\sqrt{5}$ (iii) $2+\sqrt{3}$ (iv) $2+\sqrt{5}$

(v) $5+\sqrt{7}$ (vi) $4-\sqrt{15}$ (vii) $7-\sqrt{6}$ (viii) $9+\sqrt{2}$

3. (i) If $x = 2-\sqrt{3}$, find $\frac{1}{x}$ (ii) If $x = 4-\sqrt{17}$, find $\frac{1}{x}$

(iii) If $x = \sqrt{3}+2$, find $x+\frac{1}{x}$

4. Simplify

(i) $\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}+\frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$ (ii) $\frac{1}{2+\sqrt{3}}+\frac{2}{\sqrt{5}-\sqrt{3}}+\frac{1}{2+\sqrt{5}}$

(iii) $\frac{2}{\sqrt{5}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{2}}-\frac{3}{\sqrt{5}+\sqrt{2}}$

5. (i) If $x = 2+\sqrt{3}$, find the value of $x-\frac{1}{x}$ and $\left(x-\frac{1}{x}\right)^2$

(ii) If $x = \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$, find the value of $x+\frac{1}{x}, x^2+\frac{1}{x^2}$ and $x^3+\frac{1}{x^3}$

[Hint: $a^2 + b^2 = (a + b)^2 - 2ab$ and $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$]

6. Determine the rational numbers a and b if $\frac{\sqrt{3}-1}{\sqrt{3}+1}+\frac{\sqrt{3}+1}{\sqrt{3}-1} = a+b\sqrt{3}$.

REVIEW EXERCISE 4

1. Multiple Choice Questions. Choose the correct answer.

2. Fill in the blanks.

(i) The degree of the polynomial $x^2y^2 + 3xy + y^3$ is ....

(ii) $x^2 - 4$ = ...

(iii) $x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)(\ldots\ldots\ldots)$

(iv) $2(a^2 + b^2) = (a + b)^2 + (\ldots\ldots\ldots)^2$

(v) $\left(x - \frac{1}{x}\right)^2 = \ldots\ldots\ldots$

(vi) Order of surd $\sqrt[3]{x}$ is ....

(vii) $\frac{1}{2-\sqrt{3}} = \ldots\ldots\ldots$

3.If $x + \frac{1}{x} = 3$ find (i) $x^2 + \frac{1}{x^2}$ (ii) $x^4 + \frac{1}{x^4}$

4.If $x - \frac{1}{x} = 2$ find (i) $x^2 + \frac{1}{x^2}$ (ii) $x^4 + \frac{1}{x^4}$

5.Find the value of $x^3 + y^3$ and xy if $x + y = 5$ and $x - y = 3$.

6.If $p = 2 + \sqrt{3}$ find

(i) $p + \frac{1}{p}$ (ii) $p - \frac{1}{p}$

(iii) $p^2 + \frac{1}{p^2}$ (iv) $p^2 - \frac{1}{p^2}$

7.If $q = \sqrt{5} + 2$, find

(i) $q + \frac{1}{q}$ (ii) $q - \frac{1}{q}$

(iii) $q^2 + \frac{1}{q^2}$ (iv) $q^2 - \frac{1}{q^2}$

8. Simplifying

(i) $\frac{\sqrt{a^2+2} + \sqrt{a^2-2}}{\sqrt{a^2+2} - \sqrt{a^2-2}}$ (ii) $\frac{1}{a - \sqrt{a^2 - x^2}} - \frac{1}{a + \sqrt{a^2 - x^2}}$

SUMMARY

- An algebraic expression is that in which constants or variables or both are combined by basic operations.
- Polynomial means an expression with many terms.
- Degree of polynomial means highest power of variable.

- Expression in the form $\frac{p(x)}{q(x)}$, $(q(x) \neq 0)$ is called rational expression.
- An irrational radical with rational radicand is called a surd.
- In $\sqrt[n]{x}$, n is called surd index or surd order and rational number x is called radicand.
- A surd which contains a single term is called monomial surd.
- A surd which contains sum or difference of two surds is called binomial surd.
- Conjugate surd of is $\sqrt{x} + \sqrt{y}$ defined as $\sqrt{x} - \sqrt{y}$.

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CHAPTER

5

FACTORIZATION

Animation 5.1: Factorization
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Students Learning Outcomes

After studying this unit, the students will be able to:

- Recall factorization of expressions of the following types.
  - $ka + kb + kc$
  - $ac + ad + bc + bd$
  - $a^2 \pm 2ab + b^2$
  - $a^2 - b^2$
  - $a^2 \pm 2ab + b^2 - c^2$

- Factorize the expressions of the following types.

  Type I:
  $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$

  Type II:
  $x^2 + px + q$

  Type III:
  $ax^2 + bx + c$

  Type IV:
  $$\begin{cases} (ax^2 + bx + c)(ax^2 + bx + d) + k \\ (x + a)(x + b)(x + c)(x + d) + k \\ (x + a)(x + b)(x + c)(x + d) + kx^2 \end{cases}$$

  Type V:
  $a^3 + 3a^2b + 3ab^2 + b^3$
  $a^3 - 3a^2b + 3ab^2 - b^3$

  Type VI:
  $a^3 \pm b^3$

- State and prove remainder theorem and explain through examples.
- Find Remainder (without dividing) when a polynomial is divided by a linear polynomial.
- Define zeros of a polynomial.
- State and prove Factor theorem.
- Use Factor theorem to factorize a cubic polynomial.

Introduction

Factorization plays an important role in mathematics as it helps to reduce the study of a complicated expression to the study of simpler expressions. In this unit, we will deal with different types of factorization of polynomials.

5.1 Factorization

If a polynomial $p(x)$ can be expressed as $p(x) = g(x)h(x)$, then each of the polynomials $g(x)$ and $h(x)$ is called a factor of $p(x)$. For instance, in the distributive property

$$ab + ac = a(b + c),$$

a and $(b + c)$ are factors of $(ab + ac)$.

When a polynomial has been written as a product consisting only of prime factors, then it is said to be factored completely.

(a) Factorization of the Expression of the type $ka + kb + kc$

Example 1

Factorize $5a - 5b + 5c$

Solution

$5a - 5b + 5c = 5(a - b + c)$

Example 2

Factorize $5a - 5b - 15c$

Solution

$5a - 5b - 15c = 5(a - b - 3c)$

(b) Factorization of the Expression of the type $ac + ad + bc + bd$

We can write $ac + ad + bc + bd$ as

$(ac + ad) + (bc + db)$

$= a(c + d) + b(c + d)$

$= (a + b)(c + d)$

For explanation consider the following examples.

Example 1

Factorize $3x - 3a + xy - ay$

Solution

Regrouping the terms of given polynomial

$3x + xy - 3a - ay = x(3 + y) - a(3 + y)$ (monomial factors)

$= (3 + y)(x - a)$ $(3 + y)$ is common factor

Example 2

Factorize $pqr + qr^2 - pr^2 - r^3$

Solution

The given expression $= r(pq + qr - pr - r^2)$ (r is monomial common factor)

$= r[(pq + qr) - pr - r^2]$ (grouping of terms)

$= r[q(p + r) - r(p + r)]$ (monomial factors)

$= r(p + r)(q - r)$ $(p + r)$ is common factor

(c) Factorization of the Expression of the type $a^2 \pm 2ab + b^2$

We know that

(i) $a^2 + 2ab + b^2 = (a + b)^2 = (a + b)(a + b)$

(ii) $a^2 - 2ab + b^2 = (a - b)^2 = (a - b)(a - b)$

Now consider the following examples.

Example 1

Factorize $25x^2 + 16 + 40x$.

Solution

$25x^2 + 40x + 16 = (5x)^2 + 2(5x)(4) + (4)^2$

$= (5x + 4)^2$

$= (5x + 4)(5x + 4)$

Example 2

Factorize $12x^2 - 36x + 27$

Solution

$12x^2 - 36x + 27 = 3(4x^2 - 12x + 9)$

$= 3(2x - 3)^2$

$= 3(2x - 3)(2x - 3)$

(d) Factorization of the Expression of the type $a^2 - b^2$

For explanation consider the following examples.

Example 1

Factorize (i) $4x^2 - (2y - z)^2$ (ii) $6x^4 - 96$

Solution

(i) $4x^2 - (2y - z)^2 = (2x)^2 - (2y - z)^2$

$= [2x - (2y - z)][2x + (2y - z)]$

$= (2x - 2y + z)(2x + 2y - z)$

(ii) $6x^4 - 96 = 6(x^4 - 16)$

$= 6[(x^2)^2 - (4)^2]$

$= 6(x^2 - 4)(x^2 + 4)$

$= 6[(x)^2 - (2)^2](x^2 + 4)$

$= 6(x - 2)(x + 2)(x^2 + 4)$

(e) Factorization of the Expression of the type $a^2 \pm 2ab + b^2 - c^2$

We know that

$a^2 \pm 2ab + b^2 - c^2 = (a \pm b)^2 - (c)^2 = (a \pm b - c)(a \pm b + c)$

Example 1

Factorize (i) $x^2 + 6x + 9 - 4y^2$ (ii) $1 + 2ab - a^2 - b^2$

Solution

(i) $x^2 + 6x + 9 - 4y^2 = (x + 3)^2 - (2y)^2$

$= (x + 3 + 2y)(x + 3 - 2y)$

(ii) $1 + 2ab - a^2 - b^2 = 1 - (a^2 - 2ab + b^2)$

$= (1)^2 - (a-b)^2$

$= [1-(a-b)]\,[1+(a-b)]$

$= (1-a+b)(1+a-b)$

EXERCISE 5.1

Factorize

1. (i) $2abc - 4abx + 2abd$ (ii) $9xy - 12x^2y + 18y^2$
   (iii) $-3x^2y - 3x + 9xy^2$ (iv) $5ab^2c^3 - 10a^2b^3c - 20a^3bc^2$
   (v) $3x^3y(x-3y) - 7x^2y^2(x-3y)$ (vi) $2xy^3(x^2+5) + 8xy^2(x^2+5)$
2. (i) $5ax - 3ay - 5bx + 3by$ (ii) $3xy + 2y - 12x - 8$
   (iii) $x^3 + 3xy^2 - 2x^2y - 6y^3$ (iv) $(x^2-y^2)z + (y^2-z^2)x$
3. (i) $144a^2 + 24a + 1$ (ii) $\frac{a^2}{b^2} - 2 + \frac{b^2}{a^2}$
   (iii) $(x+y)^2 - 14z(x+y) + 49z^2$ (iv) $12x^2 - 36x + 27$
4. (i) $3x^2 - 75y^2$ (ii) $x(x-1) - y(y-1)$
   (iii) $128am^2 - 242an^2$ (iv) $3x - 243x^3$
5. (i) $x^2 - y^2 - 6y - 9$ (ii) $x^2 - a^2 + 2a - 1$
   (iii) $4x^2 - y^2 - 2y - 1$ (iv) $x^2 - y^2 - 4x - 2y + 3$
   (v) $25x^2 - 10x + 1 - 36z^2$ (vi) $x^2 - y^2 - 4xz + 4z^2$

(a) Factorization of the Expression of types $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$

Factorization of such types of expression is explained in the following examples.

Example 1

Factorize $81x^4 + 36x^2y^2 + 16y^4$

Solution

$81x^4 + 36x^2y^2 + 16y^4$

$= (9x^2)^2 + 72x^2y^2 + (4y^2)^2 - 36x^2y^2$

$= (9x^2 + 4y^2)^2 - (6xy)^2$

$= (9x^2 + 4y^2 + 6xy)(9x^2 + 4y^2 - 6xy)$

$= (9x^2 + 6xy + 4y^2)\,(9x^2 - 6xy + 4y^2)$

Example 2

Factorize $9x^4 + 36y^4$

Solution

$9x^4 + 36y^4 = 9x^4 + 36y^4 + 36x^2y^2 - 36x^2y^2$

$= (3x^2)^2 + 2(3x^2)(6y^2) + (6y^2)^2 - (6xy)^2$

$= (3x^2 + 6y^2)^2 - (6xy)^2$

$= (3x^2 + 6y^2 + 6xy)(3x^2 + 6y^2 - 6xy)$

$= (3x^2 + 6xy + 6y^2)\,(3x^2 - 6xy + 6y^2)$

(b) Factorization of the Expression of the type $x^2 + px + q$

For explanation consider the following examples.

Example 1

Factorize (i) $x^2 - 7x + 12$ (ii) $x^2 + 5x - 36$

Solution

(i) $x^2 - 7x + 12$

From the factors of 12 the suitable pair of numbers is –3 and –4 since

$(-3) + (-4) = -7$ and $(-3)(-4) = 12$

Hence $x^2 - 7x + 12 = x^2 - 3x - 4x + 12$

$= x(x-3) - 4(x-3)$

$= (x-3)(x-4)$

(ii) $x^2 + 5x - 36$

From the possible factors of 36, the suitable pair is 9 and −4 because

$9 + (-4) = 5$ and $9 \times (-4) = -36$

Hence $x^2 + 5x - 36 = x^2 + 9x - 4x - 36$

$= x(x+9) - 4(x+9)$

$= (x+9)(x-4)$

(c) Factorization of the Expression of the type $ax^2 + bx + c, a \neq 0$

Let us explain the procedure of factorization by the following examples.

Example 1

Factorize (i) $9x^2 + 21x - 8$ (ii) $2x^2 - 8x - 42$ (iii) $10x^2 - 41xy + 21y^2$

Solution

(i) $9x^2 + 21x - 8$

In this case, on comparing with $ax^2 + bx + c$, $ac = (9)(-8) = -72$

From the possible factors of 72, the suitable pair of numbers (with proper sign) is 24 and –3 whose sum = 24 + (–3) = 21, (the coefficient of x)

and their product = (24) (–3) = –72 = ac

Hence $9x^2 + 21x - 8$

$= 9x^2 + 24x - 3x - 8$

$= 3x(3x + 8) - (3x + 8)$

$= (3x + 8)(3x - 1)$

(ii) $2x^2 - 8x - 42 = 2(x^2 - 4x - 21)$

Comparing $x^2 - 4x - 21$ with $ax^2 + bx + c$

we have $ac = (+1)(-21) = -21$

From the possible factors of 21, the suitable pair of numbers is –7 and +3 whose sum = –7 + 3 = – 4 and product = (–7) (3) = –21

Hence $x^2 - 4x - 21$

$= x^2 + 3x - 7x - 21$

$= x(x + 3) - 7(x + 3)$

$= (x + 3)(x - 7)$

Hence $2x^2 - 8x - 42 = 2(x^2 - 4x - 21) = 2(x + 3)(x - 7)$

(iii) $10x^2 - 41xy + 21y^2$

This type of question on factorization can also be done by the above procedures of splitting the middle term.

Here $ac = (10)(21) = 210$

Two suitable factors of 210 are –35 and –6

Their sum = –35 – 6 = –41

and product = (–35) (–6) = 210

Hence $10x^2 - 41xy + 21y^2$

$= 10x^2 - 35xy - 6xy + 21y^2$

$= 5x(2x - 7y) - 3y(2x - 7y)$

$= (2x - 7y)(5x - 3y)$

(d) Factorization of the following types of Expressions

$(ax^2 + bx + c)(ax^2 + bx + d) + k$

$(x + a)(x + b)(x + c)(x + d) + k$

$(x + a)(x + b)(x + c)(x + d) + kx^2$

We shall explain the method of factorizing these types of expressions with the help of following examples.

Example 1

Factorize $(x^2 - 4x - 5)(x^2 - 4x - 12) - 144$

Solution

$(x^2 - 4x - 5)(x^2 - 4x - 12) - 144$

Let $y = x^2 - 4x$. Then

$(y - 5)(y - 12) - 144 = y^2 - 17y - 84$

$= y^2 - 21y + 4y - 84$

$= y(y - 21) + 4(y - 21)$

$= (y - 21)(y + 4)$

$= (x^2 - 4x - 21)(x^2 - 4x + 4)$ (since $y = x^2 - 4x$)

$= (x^2 - 7x + 3x - 21)(x - 2)^2$

$= [x(x - 7) + 3(x - 7)](x - 2)^2$

$= (x - 7)(x + 3)(x - 2)(x - 2)$

Example 2

Factorize $(x + 1)(x + 2)(x + 3)(x + 4) - 120$

Solution

We observe that 1 + 4 = 2 + 3.

It suggests that we rewrite the given expression as

$[(x + 1)(x + 4)][(x + 2)(x + 3)] - 120$

$(x^2 + 5x + 4)(x^2 + 5x + 6) - 120$

Let $x^2 + 5x = y$, then

we get $(y + 4)(y + 6) - 120$

$= y^2 + 10y + 24 - 120$
$= y^2 + 10y - 96$
$= y^2 + 16y - 6y - 96$
$= y(y + 16) - 6(y + 16)$
$= (y + 16)(y - 6)$
$= (x^2 + 5x + 16)(x^2 + 5x - 6)$ since $y = x^2 + 5x$
$= (x^2 + 5x + 16)(x + 6)(x - 1)$

Example 3

Factorize $(x^2 - 5x + 6)(x^2 + 5x + 6) - 2x^2$

Solution

$(x^2 - 5x + 6)(x^2 + 5x + 6) - 2x^2$
$= [x^2 - 3x - 2x + 6][x^2 + 3x + 2x + 6] - 2x^2$
$= [x(x - 3) - 2(x - 3)][x(x + 3) + 2(x + 3)] - 2x^2$
$= [(x - 3)(x - 2)][(x + 3)(x + 2)] - 2x^2$
$= [(x - 2)(x + 2)][(x - 3)(x + 3)] - 2x^2$
$= (x^2 - 4)(x^2 - 9) - 2x^2$
$= x^4 - 13x^2 + 36 - 2x^2$
$= x^4 - 15x^2 + 36$
$= x^4 - 12x^2 - 3x^2 + 36$
$= x^2(x^2 - 12) - 3(x^2 - 12)$
$= (x^2 - 12)(x^2 - 3)$

$$= [(x)^2 - (2\sqrt{3})^2][(x)^2 - (\sqrt{3})^2]$$

$$= (x - 2\sqrt{3})(x + 2\sqrt{3})(x - \sqrt{3})(x + \sqrt{3})$$

(e) Factorization of Expressions of the following Types

$a^3 + 3a^2b + 3ab^2 + b^3$

$a^3 - 3a^2b + 3ab^2 - b^3$

For explanation consider the following examples.

Example 1

Factorize $x^3 - 8y^3 - 6x^2y + 12xy^2$

Solution

$x^3 - 8y^3 - 6x^2y + 12xy^2.$
$= (x)^3 - (2y)^3 - 3(x)^2(2y) + 3(x)(2y)^2$
$= (x)^3 - 3(x)^2(2y) + 3(x)(2y)^2 - (2y)^3$
$= (x - 2y)^3$
$= (x - 2y)(x - 2y)(x - 2y)$

(f) Factorization of Expressions of the following types $a^3 \pm b^3$

We recall the formulas,

$\boldsymbol{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$

$\boldsymbol{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$

For explanation consider the following examples.

Example 1

Factorize $27x^3 + 64y^3$

Solution

$27x^3 + 64y^3 = (3x)^3 + (4y)^3$
$= (3x + 4y)[(3x)^2 - (3x)(4y) + (4y)^2]$
$= (3x + 4y)(9x^2 - 12xy + 16y^2)$

Example 2

Factorize $1 - 125x^3$

Solution

$1 - 125x^3 = (1)^3 - (5x)^3$
$= (1 - 5x)[(1)^2 + (1)(5x) + (5x)^2]$
$= (1 - 5x)(1 + 5x + 25x^2)$

EXERCISE 5.2

Factorize

1. (i) $x^4 + \frac{1}{x^4} - 3$ (ii) $3x^4 + 12y^4$ (iii) $a^4 + 3a^2b^2 + 4b^4$

   (iv) $4x^4 + 81$ (v) $x^4 + x^2 + 25$ (vi) $x^4 + 4x^2 + 16$

2. (i) $x^2 + 14x + 48$ (ii) $x^2 - 21x + 108$

   (iii) $x^2 - 11x - 42$ (iv) $x^2 + x - 132$

3. (i) $4x^2 + 12x + 5$ (ii) $30x^2 + 7x - 15$

   (iii) $24x^2 - 65x + 21$ (iv) $5x^2 - 16x - 21$

   (v) $4x^2 - 17xy + 4y^2$ (vi) $3x^2 - 38xy - 13y^2$

   (vii) $5x^2 + 33xy - 14y^2$ (viii) $\left(5x - \frac{1}{x}\right)^2 + 4\left(5x - \frac{1}{x}\right) + 4, x \neq 0$

4. (i) $(x^2 + 5x + 4)(x^2 + 5x + 6) - 3$

   (ii) $(x^2 - 4x)(x^2 - 4x - 1) - 20$

   (iii) $(x + 2)(x + 3)(x + 4)(x + 5) - 15$

   (iv) $(x + 4)(x - 5)(x + 6)(x - 7) - 504$

   (v) $(x + 1)(x + 2)(x + 3)(x + 6) - 3x^2$

5. (i) $x^3 + 48x - 12x^2 - 64$ (ii) $8x^3 + 60x^2 + 150x + 125$

   (iii) $x^3 - 18x^2 + 108x - 216$ (iv) $8x^3 - 125y^3 - 60x^2y + 150xy^2$

6. (i) $27 + 8x^3$ (ii) $125x^3 - 216y^3$

   (iii) $64x^3 + 27y^3$ (iv) $8x^3 + 125y^3$

5.2 Remainder Theorem and Factor Theorem

5.2.1 Remainder Theorem

If a polynomial $p(x)$ is divided by a linear divisor $(x - a)$, then the remainder is $p(a)$.

Proof

Let $q(x)$ be the quotient obtained after dividing $p(x)$ by $(x - a)$. But the divisor $(x - a)$ is linear. So the remainder must be of degree zero i.e., a non-zero constant, say R. Consequently, by division Algorithm we may write

$$p(x) = (x - a)\,q(x) + R$$

This is an identity in x and so is true for all real numbers x. In particular, it is true for $x = a$. Therefore,

$$p(a) = (a - a)\,q(a) + R = 0 + R = R$$

i.e., $p(a)$ = the remainder. Hence the theorem.

Note: Similarly, if the divisor is $(ax - b)$, we have

$$p(x) = (ax - b)\,q(x) + R$$

Substituting $x = \frac{a}{b}$ so that $ax - b = 0$, we obtain

$$p\left(\frac{b}{a}\right) = 0 \,.\, q\left(\frac{b}{a}\right) + R = 0 - R = R$$

Thus if the divisor is linear, the above theorem provides an efficient way of finding the remainder without being involved in the process of long division.

5.2.2 To find Remainder (without dividing) when a polynomial is divided by a Linear Polynomial

Example 1

Find the remainder when $9x^2 - 6x + 2$ is divided by

(i) $x - 3$ (ii) $x + 3$ (iii) $3x + 1$ (iv) x

Solution

Let $p(x) = 9x^2 - 6x + 2$

(i) When $p(x)$ is divided by $x - 3$, by Remainder Theorem, the remainder is

$$R = p(3) = 9(3)^2 - 6(3) + 2 = 65$$

(ii) When $p(x)$ is divided by $x + 3 = x - (-3)$, the remainder is

$$R = p(-3) = 9(-3)^2 - 6(-3) + 2 = 101$$

(iii) When $p(x)$ is divided by $3x + 1$, the remainder is

$$R = p\left(-\frac{1}{3}\right) = 9\left(-\frac{1}{3}\right)^2 - 6\left(-\frac{1}{3}\right) + 2 = 5$$

(v) When $p(x)$ is divided by x, the remainder is

$$R = p(0) = 9(0)^2 - 6(0) + 2 = 2$$

Example 2

Find the value of k if the expression $x^3 + kx^2 + 3x - 4$ leaves a remainder of –2 when divided by $x + 2$.

Solution

Let $p(x) = x^3 + kx^2 + 3x - 4$

By the Remainder Theorem, when $p(x)$ is divided by $x + 2 = x - (-2)$, the remainder is

$$p(-2) = (-2)^3 + k(-2)^2 + 3(-2) - 4.$$
$$= -8 + 4k - 6 - 4$$
$$= 4k - 18$$

By the given condition, we have

$$p(-2) = -2 \quad \Rightarrow \quad 4k - 18 = -2 \quad \Rightarrow \quad k = 4$$

5.2.3 Zero of a Polynomial

Definition

If a specific number $x = a$ is substituted for the variable x in a polynomial $p(x)$ so that the value $p(a)$ is zero, then $x = a$ is called a zero of the polynomial $p(x)$.

A very useful consequence of the remainder theorem is what is known as the factor theorem.

5.2.4 Factor Theorem

The polynomial $(x - a)$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0$.

Proof

Let $q(x)$ be the quotient and R the remainder when a polynomial $p(x)$ is divided by $(x - a)$. Then by division Algorithm,

$$p(x) = (x - a)\, q(x) + R$$

By the Remainder Theorem, $R = p(a)$.

Hence $p(x) = (x - a)\, q(x) + p(a)$

(i) Now if $p(a) = 0$, then $p(x) = (x - a)\, q(x)$
i.e., $(x - a)$ is a factor of $p(x)$

(ii) Conversely, if $(x - a)$ is a factor of $p(x)$, then the remainder upon dividing p(x) by $(x - a)$ must be zero i.e., $p(a) = 0$
This completes the proof.

Note: The Factor Theorem can also be stated as, "$(x - a)$ is a factor of p(x) if and only if $x = a$ is a solution of the equation p(x) = 0".

The Factor Theorem helps us to find factors of polynomials because it determines whether a given linear polynomial $(x - a)$ is a factor of p(x). All we need is to check whether p(a) = 0.

Example 1

Determine if $(x - 2)$ is a factor of $x^3 - 4x^2 + 3x + 2$.

Solution

For convenience, let

$$p(x) = x^3 - 4x^2 + 3x + 2$$

Then the remainder for $(x - 2)$ is

$$p(2) = (2)^3 - 4(2)^2 + 3(2) + 2$$
$$= 8 - 16 + 6 + 2 = 0$$

Hence by Factor Theorem, $(x - 2)$ is a factor of the polynomial $p(x)$.

Example 2

Find a polynomial $p(x)$ of degree 3 that has 2, –1, and 3 as zeros (i.e., roots).

Solution

Since $x = 2, -1, 3$ are roots of $p(x) = 0$

So by Factor Theorem $(x - 2)$, $(x + 1)$ and $(x - 3)$ are the factors of $p(x)$.

Thus $p(x) = a(x - 2)(x + 1)(x - 3)$

where any non-zero value can be assigned to a.

Taking $a = 1$, we get

$$p(x) = (x - 2)(x + 1)(x - 3)$$
$$= x^3 - 4x^2 + x + 6 \quad \text{as the required polynomial.}$$

EXERCISE 5.3

1. Use the remainder theorem to find the remainder when
   (i) $3x^3 - 10x^2 + 13x - 6$ is divided by $(x - 2)$
   (ii) $4x^3 - 4x + 3$ is divided by $(2x - 1)$
   (iii) $6x^4 + 2x^3 - x + 2$ is divided by $(x + 2)$
   (iv) $(2x - 1)^3 + 6(3 + 4x)^2 - 10$ is divided by $(2x + 1)$
   (v) $x^3 - 3x^2 + 4x - 14$ is divided by $(x + 2)$
2. (i) If $(x + 2)$ is a factor of $3x^2 - 4kx - 4k^2$, then find the value(s) of k.
   (ii) If $(x - 1)$ is a factor of $x^3 - kx^2 + 11x - 6$, then find the value of k.
3. Without actual long division determine whether
   (i) $(x - 2)$ and $(x - 3)$ are factors of $p(x) = x^3 - 12x^2 + 44x - 48$.
   (ii) $(x - 2)$, $(x + 3)$ and $(x - 4)$ are factors of $q(x) = x^3 + 2x^2 - 5x - 6$.
4. For what value of m is the polynomial $p(x) = 4x^3 - 7x^2 + 6x - 3m$ exactly divisible by $x + 2$?
5. Determine the value of k if $p(x) = kx^3 + 4x^2 + 3x - 4$ and $q(x) = x^3 - 4x + k$ leaves the same remainder when divided by $(x - 3)$.
6. The remainder after dividing the polynomial $p(x) = x^3+ax^2+7$ by $(x + 1)$ is $2b$. Calculate the value of a and b if this expression leaves a remainder of $(b + 5)$ on being divided by $(x - 2)$.
7. The polynomial $x^3 + lx^2 + mx + 24$ has a factor $(x + 4)$ and it leaves a remainder of 36 when divided by $(x - 2)$. Find the values of l and m.
8. The expression $lx^3 + mx^2 - 4$ leaves remainder of -3 and 12 when divided by $(x - 1)$ and $(x + 2)$ respectively. Calculate the values of l and m.
9. The expression $ax^3 - 9x^2 + bx + 3a$ is exactly divisible by $x^2 - 5x + 6$. Find the values of a and b.

5.3 Factorization of a Cubic Polynomial

We can use Factor Theorem to factorize a cubic polynomial as explained below. This is a convenient method particularly for factorization of a cubic polynomial. We state (without proof) a very useful Theorem.

Rational Root Theorem

Let $a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0, \quad a_0 \neq 0$
be a polynomial equation of degree n with integral coefficients. If p/q is a rational root (expressed in lowest terms) of the equation, then p is a factor of the constant term a_n and q is a factor of the leading coefficient a_0.

Example 1

Factorize the polynomial $x^3 - 4x^2 + x + 6$, by using Factor Theorem.

Solution

We have $P(x) = x^3 - 4x^2 + x + 6$.

Possible factors of the constant term $p = 6$ are ±1, ±2, ±3 and ±6 and of leading coefficient $q = 1$ are ±1. Thus the expected zeros (or roots) of $P(x) = 0$ are $\frac{p}{q} = \pm1, \pm2, \pm3$ and ±6. If $x = a$ is a zero of $P(x)$, then $(x - a)$ will be a factor.

We use the hit and trial method to find zeros of $P(x)$. Let us try $x = 1$.

Now $P(1) = (1)^3 - 4(1)^2 + 1 + 6$
$= 1 - 4 + 1 + 6 = 4 \neq 0$

Hence $x = 1$ is not a zero of $P(x)$.

Again $P(-1) = (-1)^3 - 4(-1)^2 - 1 + 6$
$= -1 - 4 - 1 + 6 = 0$

Hence $x = -1$ is a zero of $P(x)$ and therefore,
$x - (-1) = (x + 1)$ is a factor of $P(x)$.

Now $P(2) = (2)^3 - 4(2)^2 + 2 + 6$
$= 8 - 16 + 2 + 6 = 0 \Rightarrow x = 2$ is a root.

Hence $(x - 2)$ is also a factor of $P(x)$.

Similarly $P(3) = (3)^3 - 4(3)^2 + 3 + 6$
$= 27 - 36 + 3 + 6 = 0 \Rightarrow x = 3$ is a zero of $P(x)$.

Hence $(x - 3)$ is the third factor of $P(x)$.

Thus the factorized form of

$$P(x) = x^3 - 4x^2 + x + 6$$

is $P(x) = (x + 1)(x - 2)(x - 3)$

EXERCISE 5.4

Factorize each of the following cubic polynomials by factor theorem.

1.	$x^3 - 2x^2 - x + 2$	2.	$x^3 - x^2 - 22x + 40$	3.	$x^3 - 6x^2 + 3x + 10$
4.	$x^3 + x^2 - 10x + 8$	5.	$x^3 - 2x^2 - 5x + 6$	6.	$x^3 + 5x^2 - 2x - 24$
7.	$3x^3 - x^2 - 12x + 4$	8.	$2x^3 + x^2 - 2x - 1$		

REVIEW EXERCISE 5

1. Multiple Choice Questions. Choose the correct answer.

2. Completion Items. Fill in the blanks.
   (i) $x^2 + 5x + 6 =$ ....
   (ii) $4a^2 - 16 =$ ....
   (iii) $4a^2 + 4ab +$ (.........) is a complete square
   (iv) $\frac{x^2}{y^2} - 2 + \frac{y^2}{x^2} =$ ......
   (v) $(x + y)(x^2 - xy + y^2) =$ ....
   (vi) Factored form of $x^4 - 16$ is ....
   (vii) If $x - 2$ is factor of $p(x) = x^2 + 2kx + 8$, then $k =$ ....

3. Factorize the following.
   (i) $x^2 + 8x + 16 - 4y^2$ (ii) $4x^2 - 16y^2$
   (iii) $9x^2 + 27x + 8$ (iv) $1 - 64z^3$
   (v) $8x^3 - \frac{1}{27y^3}$ (vi) $2y^2 + 5y - 3$
   (vii) $x^3 + x^2 - 4x - 4$ (viii) $25m^2n^2 + 10mn + 1$
   (ix) $1 - 12pq + 36p^2q^2$

SUMMARY

- If a polynomial is expressed as a product of other polynomials, then each polynomial in the product is called a factor of the original polynomial.
- The process of expressing an algebraic expression in terms of its factors is called factorization. We learned to factorize expressions of the following types:
  - $ka + kb + kc$
  - $ac + ad + bc + bd$
  - $a^2 \pm 2ab + b^2$
  - $a^2 - b^2$
  - $(a^2 \pm 2ab + b^2) - c^2$
  - $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$
  - $x^2 + px + q$ • $ax^2 + bx + c$
  - $(ax^2 + bx + c)(ax^2 + bx + d) + k$
  - $(x + a)(x + b)(x + c)(x + d) + k$
  - $(x + a)(x + b)(x + c)(x + d) + kx^2$
  - $a^3 + 3a^2b + 3ab^2 + b^3$
  - $a^3 - 3a^2b + 3ab^2 - b^3$
  - $a^3 \pm b^3$
- If a polynomial $p(x)$ is divided by a linear divisor $(x - a)$, then the remainder is $p(a)$.
- If a specific number $x = a$ is substituted for the variable x in a polynomial $p(x)$ so that the value $p(a)$ is zero, then $x = a$ is called a zero of the polynomial $p(x)$.
- The polynomial $(x - a)$ is a factors of the polynomial $p(x)$ if and only if $p(a) = 0$. Factor theorem has been used to factorize cubic polynomials.

CHAPTER

6

ALGEBRAIC MANIPULATION

Animation 6.1: Algebraic Manipulation
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Students Learning Outcomes

After studying this unit, the students will be able to:

- Find Highest Common Factor and Least Common Multiple of algebraic expressions.
- Use factor or division method to determine Highest Common Factor and Least Common Multiple.
- Know the relationship between H.C.F. and L.C.M.
- Solve real life problems related to H.C.F. and L.C.M.
- Use Highest Common Factor and Least Common Multiple to reduce fractional expressions involving $+, -, \times, \div$.
- Find square root of algebraic expressions by factorization and division.

Introduction

In this unit we will first deal with finding H.C.F. and L.C.M. of algebraic expressions by factorization and long division. Then by using H.C.F. and L.C.M. we will simplify fractional expressions. Toward the end of the unit finding square root of algebraic expression by factorization and division will be discussed.

6.1 Highest Common Factor (H.C.F.) and Least Common Multiple (L.C.M.) of Algebraic Expressions

6.1.1 (a) Highest Common Factor (H.C.F.)

If two or more algebraic expressions are given, then their common factor of highest power is called the H.C.F. of the expressions.

(b) Least Common Multiple (L.C.M.)

If an algebraic expression $p(x)$ is exactly divisible by two or more expressions, then $p(x)$ is called the Common Multiple of the given expressions. The Least Common Multiple (L.C.M.) is the product of common factors together with non-common factors of the given expressions.

6.1.2 (a) Finding H.C.F.

We can find H. C. F. of given expressions by the following two methods.

(i) By Factorization **(ii) By Division**

Sometimes it is difficult to find factors of given expressions. In that case, method of division can be used to find H. C. F. We consider some examples to explain these two methods.

(i) H.C.F. by Factorization

Example

Find the H. C. F. of the following polynomials.

$$x^2 - 4,\ x^2 + 4x + 4,\ 2x^2 + x - 6$$

Solution

By factorization,

$x^2 - 4 = (x + 2)(x - 2)$

$x^2 + 4x + 4 = (x + 2)^2$

$2x^2 + x - 6 = 2x^2 + 4x - 3x - 6 = 2x(x + 2) - 3(x + 2)$

$= (x + 2)(2x - 3)$

Hence, H. C. F. $= x + 2$

(ii) H.C.F. by Division

Example

Use division method to find the H. C. F. of the polynomials

$$p(x) = x^3 - 7x^2 + 14x - 8 \text{ and } q(x) = x^3 - 7x + 6$$

Solution

$$\begin{array}{r|l} & \quad 1 \\ \hline x^3 - 7x + 6 & x^3 - 7x^2 + 14x - 8 \\ & \pm x^3 \qquad \mp 7x \pm 6 \\ \hline & -7x^2 + 21x - 14 \end{array}$$

Here the remainder can be factorized as

$$-7x^2 + 21x - 14 = -7(x^2 - 3x + 2)$$

We ignore –7 because it is not common to both the given polynomials and consider $x^2 - 3x + 2$.

$$\begin{array}{r|l} & \quad x + 3 \\ \hline x^2 - 3x + 2 & x^3 + 0x^2 - 7x + 6 \\ & \pm x^3 \mp 3x^2 \pm 2x \\ \hline & 3x^2 - 9x + 6 \\ & \pm 3x^2 \mp 9x \pm 6 \\ \hline & \quad 0 \end{array}$$

Hence H. C. F. of $p(x)$ and $q(x)$ is $x^2 - 3x + 2$

Observe that

(i) In finding H. C. F. by division, if required, any expression can be multiplied by a suitable integer to avoid fraction.

(ii) In case we are given three polynomials, then as a first step we find H.C.F. of any two of them and then find the H.C.F. of this H.C.F. and the third polynomial.

(b) L.C.M. by Factorization

Working Rule to find L.C.M. of given Algebraic Expressions

(i) Factorize the given expressions completely i.e., to simplest form.

(ii) Then the L.C.M. is obtained by taking the product of each factor appearing in any of the given expressions, raised to the highest power with which that factor appears.

Example

Find the L.C.M. of $p(x) = 12(x^3 - y^3)$ and $q(x) = 8(x^3 - xy^2)$

Solution

By prime factorization of the given expressions, we have

$p(x) = 12(x^3 - y^3) = 2^2 \times 3 \times (x - y)(x^2 + xy + y^2)$

and $q(x) = 8(x^3 - xy^2) = 8x(x^2 - y^2) = 2^3x(x + y)(x - y)$

Hence L.C.M. of $p(x)$ and $q(x)$,

$2^3 \times 3 \times x(x + y)(x - y)(x^2 + xy + y^2) = 24x(x + y)(x^3 - y^3)$

6.1.3 Relation between H.C.F. and L.C.M.

Example

By factorization, find (i) H.C.F. (ii) L.C.M. of $p(x) = 12(x^5 - x^4)$ and $q(x) = 8(x^4 - 3x^3 + 3x^2)$. Establish a relation between $p(x)$, $q(x)$ and H.C.F. and L.C.M. of the expressions $p(x)$ and $q(x)$.

Solution

Firstly, let us factorize completely the given expressions $p(x)$ and $q(x)$ into irreducible factors. We have

$p(x) = 12(x^5 - x^4) = 12x^4(x - 1) = 2^2 \times 3 \times x^4(x - 1)$

and $q(x) = 8(x^4 - 3x^3 + 2x^2) = 8x^2(x^2 - 3x + 2) = 2^3 x^2(x - 1)(x - 2)$

H.C.F. of $p(x)$ and $q(x) = 2^2 x^2 (x - 1) = 4x^2(x - 1)$

L.C.M. of $p(x)$ and $q(x) = 2^3 \times 3 \times x^4(x - 1)(x - 2)$

Observe that

$p(x)q(x) = 12x^4(x - 1) - 8x^2(x - 1)(x - 2)$

$= 96x^6(x - 1)^2(x - 2)$ ...... (i)

and (L.C.M.) (H.C.F.)

$= [2^3 \times 3 \times x^4(x - 1)(x - 2)][4x^2(x - 1)]$

$= [24x^4(x - 1)(x - 2)][4x^2(x - 1)]$

$= 96x^6(x - 1)^2(x - 2)$ ...... (ii)

From (i) and (ii) it is clear that

$$\boxed{\text{L.C.M.} \times \text{H.C.F.} = p(x) \times q(x)}$$

Hence, if $p(x)$, $q(x)$ and one of H.C.F. or L.C.M. are known, we can find the unknown by the formulae,

I. $\text{L.C.M} = \dfrac{p(x) \times q(x)}{\text{H.C.F}}$ or $\text{H.C.F} = \dfrac{p(x) \times q(x)}{\text{L.C.M}}$

II. If L.C.M., H.C.F. and one of $p(x)$ or $q(x)$ are known, then

$$p(x) = \frac{\text{L.C.M} \times \text{H.C.F.}}{q(x)},$$

$$q(x) = \frac{\text{L.C.M} \times \text{H.C.F.}}{p(x)}$$

Note: L.C.M. and H.C.F. are unique except for a factor of (–1).

Example 1

Find H.C.F. of the polynomials,

$p(x) = 20(2x^3 + 3x^2 - 2x)$

$q(x) = 9(5x^4 + 40x)$

Then using the above formula (I) find the L.C.M. of $p(x)$ and $q(x)$.

Solution

We have

$p(x) = 20(2x^3 + 3x^2 - 2x) = 20x\,(2x^2 + 3x - 2)$

$= 20x(2x^2 + 4x - x - 2) = 20x[2x(x + 2) - (x + 2)]$

$= 20x\,(x + 2)\,(2x - 1) = 2^2 \times 5 \times x\,(x + 2)\,(2x - 1)$

$q(x) = 9(5x^4 + 40x) = 45x(x^3 + 8)$

$= 45x\,(x + 2)\,(x^2 - 2x + 4) = 5 \times 3^2 \times x\,(x + 2)\,(x^2 - 2x + 4)$

Thus H.C.F. of p(x) and q(x) is

$= 5x\,(x + 2)$

Now, using the formula $\text{L.C.M} = \dfrac{p(x) \times q(x)}{\text{H.C.F}}$

we obtain

$$\text{L.C.M} = \frac{2^2 \times 5 \times x(x+2)(2x-1) \times 5 \times 3^2 \times x(x+2)(x^2-2x+4)}{5x(x+2)}$$

$= 4 \times 5 \times 9 \times x\,(x + 2)\,(2x - 1)\,(x^2 - 2x + 4)$

$= 180x\,(x + 2)\,(2x - 1)\,(x^2 - 2x + 4)$

Example 2

Find the L.C.M. of

$p(x) = 6x^3 - 7x^2 - 27x + 8$ and $q(x) = 6x^3 + 17x^2 + 9x - 4$

Solution

We have, by long division,

$$\begin{array}{r|l} & \quad 1 \\ 6x^3 - 7x^2 - 27x + 8 & \; 6x^3 + 17x^2 + 9x - 4 \\ & \; 6x^3 - 7x^2 - 27x + 8 \\ & \; - \quad + \quad + \quad - \\ \hline & \quad 24x^2 + 36x - 12 \end{array}$$

But the remainder $24x^2 + 36x - 12$

$= 12(2x^2 + 3x - 1)$

Thus, ignoring 12, we have

$$\begin{array}{r|l} & \quad 3x - 8 \\ 2x^2 + 3x - 1 & \; 6x^3 - 7x^2 - 27x + 8 \\ & \; 6x^3 + 9x^2 - 3x \\ & \; - \quad - \quad + \\ \hline & \quad -16x^2 - 24x + 8 \\ & \quad -16x^2 - 24x + 8 \\ & \quad + \quad + \quad - \\ \hline & \qquad 0 \end{array}$$

Hence H.C.F. of $p(x)$ and $q(x)$ is $= 2x^2 + 3x - 1$

By using the formula, we have

$$\text{L.C.M} = \frac{p(x) \times q(x)}{\text{H.C.F}}$$

$$= \frac{(6x^3 - 7x^2 - 27x + 8)(6x^3 + 17x^2 + 9x - 4)}{2x^2 + 3x - 1}$$

$$= \frac{6x^3 - 7x^2 - 27x + 8}{2x^2 + 3x - 1} \times (6x^3 + 17x^2 + 9x - 4)$$

$= (3x - 8)(6x^3 + 17x^2 + 9x - 4)$

6.1.4 Application of H.C.F. and L.C.M.

Example

The sum of two numbers is 120 and their H.C.F. is 12. Find the numbers.

Solution

Let the numbers be $12x$ and $12y$, where x, y are numbers prime to each other.

Then $12x + 12y = 120$

i.e., $x + y = 10$

Thus we have to find two numbers whose sum is 10. The possible such pairs of numbers are (1, 9), (2, 8), (3, 7), (4, 6), (5, 5)

The pairs of numbers which are prime to each other are (1, 9) and (3, 7)

Thus the required numbers are

$1 \times 12, 9 \times 12; 3 \times 12, 7 \times 12$

i.e., 12, 108 and 36, 84.

EXERCISE 6.1

1. Find the H.C.F. of the following expressions.
   (i) $39x^7y^3z$ and $91x^5y^6z^7$ (ii) $102xy^2z$, $85x^2yz$ and $187xyz^2$
2. Find the H.C.F. of the following expressions by factorization.
   (i) $x^2 + 5x + 6$, $x^2 - 4x - 12$
   (ii) $x^3 - 27$, $x^2 + 6x - 27$, $2x^2 - 18$
   (iii) $x^3 - 2x^2 + x$, $x^2 + 2x - 3$, $x^2 + 3x - 4$
   (iv) $18(x^3 - 9x^2 + 8x)$, $24(x^2 - 3x + 2)$
   (v) $36(3x^4 + 5x^3 - 2x^2)$, $54(27x^4 - x)$
3. Find the H.C.F. of the following by division method.
   (i) $x^3 + 3x^2 - 16x + 12$, $x^3 + x^2 - 10x + 8$
   (ii) $x^4 + x^3 - 2x^2 + x - 3$, $5x^3 + 3x^2 - 17x + 6$
   (iii) $2x^5 - 4x^4 - 6x$, $x^5 + x^4 - 3x^3 - 3x^2$
4. Find the L.C.M. of the following expressions.
   (i) $39x^7y^3z$ and $91x^5y^6z^7$ (ii) $102xy^2z$, $85x^2yz$ and $187xyz^2$
5. Find the L.C.M. of the following expressions by factorization.
   (i) $x^2 - 25x + 100$ and $x^2 - x - 20$
   (ii) $x^2 + 4x + 4$, $x^2 - 4$, $2x^2 + x - 6$
   (iii) $2(x^4 - y^4)$, $3(x^3 + 2x^2y - xy^2 - 2y^3)$
   (iv) $4(x^4 - 1)$, $6(x^3 - x^2 - x + 1)$
6. For what value of k is $(x + 4)$ the H.C.F. of $x^2 + x - (2k + 2)$ and $2x^2 + kx - 12$?
7. If $(x+ 3)(x - 2)$ is the H.C.F. of $p(x) = (x + 3)(2x^2 - 3x + k)$ and $q(x) = (x - 2)(3x^2 + 7x - 1)$, find k and l.
8. The L.C.M. and H.C.F. of two polynomials $p(x)$ and $q(x)$ are $2(x^4 - 1)$ and $(x + 1)(x^2 + 1)$ respectively.If $p(x) = x^3 + x^2 + x + 1$, find $q(x)$.
9. Let $p(x) = 10(x^2 - 9)(x^2 - 3x + 2)$ and $q(x) = 10x(x + 3)(x - 1)^2$. If the H.C.F. of $p(x)$, $q(x)$ is $10(x + 3)(x - 1)$, find their L.C.M.
10. Let the product of L.C.M and H.C.F of two polynomials be $(x + 3)^2 (x - 2)(x + 5)$. If one polynomial is $(x + 3)(x - 2)$ and the second polynomial is $x^2 + kx + 15$, find the value of k.
11. Waqas wishes to distribute 128 bananas and also 176 apples equally among a certain number of children. Find the highest number of children who can get the fruit in this way.

6.2 Basic Operations on Algebraic Fractions

We shall now carryout the operations of addition, difference, product and division on algebraic fractions by giving some examples. We assume that all fractions are defined.

Example 1

Simplify $\frac{x+3}{x^2-3x+2} + \frac{x+2}{x^2-4x+3} + \frac{x+1}{x^2-5x+6}$, $x \neq 1, 2, 3$

Solution

$$\frac{x+3}{x^2-3x+2} + \frac{x+2}{x^2-4x+3} + \frac{x+1}{x^2-5x+6}$$

$$= \frac{x+3}{x^2-2x-x+2} + \frac{x+2}{x^2-3x-x+3} + \frac{x+1}{x^2-3x-2x+6}$$

$$= \frac{x+3}{x(x-2)-1(x-2)} + \frac{x+2}{x(x-3)-1(x-3)} + \frac{x+1}{x(x-3)-2(x-3)}$$

$$= \frac{x+3}{(x-2)(x-1)} + \frac{x+2}{(x-3)(x-1)} + \frac{x+1}{(x-3)(x-2)}$$

$$= \frac{(x+3)(x-3)+(x+2)(x-2)+(x+1)(x-1)}{(x-1)(x-2)(x-3)}$$

$$= \frac{x^2-9+x^2-4+x^2-1}{(x-1)(x-2)(x-3)}$$

$$= \frac{3x^2-14}{(x-1)(x-2)(x-3)}$$

Example 2

Express the product $\frac{x^3-8}{x^2-4} \times \frac{x^2+6x+8}{x^2-2x+1}$ as an algebraic expression reduced to lowest forms, $x \neq 2, -2, 1$

Solution

By factorizing completely, we have

$$\frac{x^3-8}{x^2-4} \times \frac{x^2+6x+8}{x^2-2x+1}$$

$$= \frac{(x-2)(x^2+2x+4) \times (x+2)(x+4)}{(x-2)(x+2) \times (x-1)^2} \quad \text{....... (i)}$$

Now the factors of numerator are $(x-2)$, (x^2+2x+4), $(x+2)$ and $(x+4)$ and the factors of denominator are

$(x-2)$, $(x+2)$ and $(x-1)^2$.

Therefore, their H.C.F. is $(x-2) \times (x+2)$.

By cancelling H.C.F. i.e., $(x-2)(x+2)$ from (I), we get the simplified form of given product as the fraction $\frac{(x^2+2x+4)(x+4)}{(x-1)^2}$

Example 3

Divide $\frac{(x^2+x+1)}{(x^2-9)}$ by $\frac{x^3-1}{(x^2-4x+3)}$ and simplify by reducing to lowest forms.

Solution

We have $\frac{(x^2+x+1)}{(x^2-9)} \div \frac{x^3-1}{(x^2-4x+3)}$

$$= \frac{(x^2+x+1)}{(x^2-9)} \times \frac{(x^2-4x+3)}{(x^3-1)} \quad \text{....... (inverting)}$$

$$= \frac{(x^2+x+1)(x^2-x-3x+3)}{(x^2-9)(x^3-1)} \quad \text{... (splitting the middle term)}$$

$$= \frac{(x^2+x+1)(x-3)(x-1)}{(x+3)(x-3)(x-1)(x^2+x+1)} = \frac{1}{x+3}, \; x \neq -3$$

EXERCISE 6.2

Simplify each of the following as a rational expression.

1. $\frac{x^2-x-6}{x^2-9} + \frac{x^2+2x-24}{x^2-x-12}$

2. $\left[\frac{x+1}{x-1} - \frac{x-1}{x+1} - \frac{4x}{x^2+1}\right] + \frac{4x}{x^4-1}$

3. $\frac{1}{x^2-8x+15} + \frac{1}{x^2-4x+3} - \frac{2}{x^2-6x+5}$

4. $\frac{(x+2)(x+3)}{x^2-9} + \frac{(x+2)(2x^2-32)}{(x-4)(x^2-x-6)}$

5. $\frac{x+3}{2x^2+9x+9} + \frac{1}{2(2x-3)} - \frac{4x}{4x^2-9}$

6. $A - \frac{1}{A}$, where $A = \frac{a+1}{a-1}$

7. $\left[\frac{x-1}{x-2} + \frac{2}{2-x}\right] - \left[\frac{x+1}{x+2} + \frac{4}{4-x^2}\right]$

8. What rational expression should be subtracted from $\frac{2x^2+2x-7}{x^2+x-6}$ to get $\frac{x-1}{x-2}$?

Perform the indicated operations and simplify to the lowest form.

9. $\dfrac{x^2+x-6}{x^2-x-6}\times\dfrac{x^2-4}{x^2-9}$

10. $\dfrac{x^3-8}{x^2-4}\times\dfrac{x^2+6x+8}{x^2-2x+1}$

11. $\dfrac{x^4-8x}{2x^2+5x-3}\times\dfrac{2x-1}{x^2+2x+4}\times\dfrac{x+3}{x^2-2x}$

12. $\dfrac{2y^2+7y-4}{3y^2-13y+4}\div\dfrac{4y^2-1}{6y^2+y-1}$

13. $\left[\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}\right]\div\left[\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right]$

6.3 Square Root of Algebraic Expression

Square Root

As with numbers define the square root of given expression $p(x)$ as another expression $q(x)$ such that $q(x)\,.\,q(x) = p(x)$.

As $5 \times 5 = 25$, so square root of 25 is 5.

It means we can find square root of the expression $p(x)$ if it can be expressed as a perfect square.

In this section we shall find square root of an algebraic expression

(i) **by factorization** (ii) **by division**

(i) By Factorization

First we find the square root by factorization.

Example 1

Use factorization to find the square root of the expression

$$4x^2 - 12x + 9$$

Solution

We have, $4x^2 - 12x + 9$

$= 4x^2 - 6x - 6x + 9 = 2x(2x - 3) - 3(2x - 3)$

$= (2x - 3)(2x - 3) = (2x - 3)^2$

Hence $\sqrt{4x^2-12x+9}$

$= \pm(2x - 3)$

Example 2

Find the square root of $x^2+\dfrac{1}{x^2}+12\left(x+\dfrac{1}{x}\right)+38,\ x\neq 0$

Solution

We have $x^2+\dfrac{1}{x^2}+12\left(x+\dfrac{1}{x}\right)+38$

$=x^2+\dfrac{1}{x^2}+2+12\left(x+\dfrac{1}{x}\right)+36$, (adding and subtracting 2)

$=\left(x+\dfrac{1}{x}\right)^2+2\left(x+\dfrac{1}{x}\right)(6)+(6)^2$

$=\left[\pm\left(x+\dfrac{1}{x}+6\right)\right]^2$; since $a^2 + 2ab + b^2 = (a + b)^2$

Hence the required square root is $\pm\left(x+\dfrac{1}{x}+6\right)$

(ii) By Division

When it is difficult to convert the given expression into a perfect square by factorization, we use the method of actual division to find its square root. The method is similar to the division method of finding square root of numbers.

Note that

We first write the given expression in descending order of powers of x.

Example 1

Find the square root of $4x^4 + 12x^3 + x^2 - 12x + 4$

Solution

We note that the given expression is already in descending order. Now the square root of the first term i.e., $\sqrt{4x^4} = 2x^2$. So the first term of the divisor and quotient will be $2x^2$ in the first step. At each successive step, the remaining terms will be brought down.

$$
\begin{array}{r|l}
 & 2x^2+3x-2 \\
\hline
2x^2 & 4x^4+12x^3+x^2-12x+4 \\
 & \pm 4x^4 \\
\hline
4x^2+3x & 12x^3+x^2-12x+4 \\
 & \pm 12x^3 \pm 9x^2 \\
\hline
4x^2+6x-2 & -8x^2-12x+4 \\
 & \mp 8x^2 \mp 12x \pm 4 \\
\hline
 & 0
\end{array}
$$

Thus square root of given expression is $\pm(2x^2 + 3x - 2)$

Example 2

Find the square root of the expression

$$4\frac{x^2}{y^2}+8\frac{x}{y}+16+12\frac{y}{x}+9\frac{y^2}{x^2}$$

Solution

We note that the given expression is in descending powers of x.

Now $\sqrt{4\frac{x^2}{y^2}}=2\frac{x}{y}$. So proceeding as usual, we have

$$
\begin{array}{r|l}
 & 2\frac{x}{y}+2+3\frac{y}{x} \\
\hline
2\frac{x}{y} & 4\frac{x^2}{y^2}+8\frac{x}{y}+16+12\frac{y}{x}+9\frac{y^2}{x^2} \\
 & \pm 4\frac{x^2}{y^2} \\
\hline
4\frac{x}{y}+2 & 8\frac{x}{y}+16 \\
 & \pm 8\frac{x}{y} \pm 4 \\
\hline
4\frac{x}{y}+4+3\frac{y}{x} & 12+12\frac{y}{x}+9\frac{y^2}{x^2} \\
 & \pm 12 \pm 12\frac{y}{x} \pm 9\frac{y^2}{x^2} \\
\hline
 & 0
\end{array}
$$

Hence the square root of given expression is

$$\pm\left(2\frac{x}{y}+2+3\frac{y}{x}\right)$$

Example 3

To make the expression $x^4 - 10x^3 + 33x^2 - 42x + 20$ a perfect square,

(i) what should be added to it?

(ii) what should be subtract from it?

(iii) what should be the values of x?

Solution

$$
\begin{array}{r|l}
 & x^2-5x+4 \\
\hline
x^2 & x^4-10x^3+33x^2-42x+20 \\
 & \pm x^4 \\
\hline
2x^2-5x & -10x^3+33x^2 \\
 & \mp 10x^3 \pm 25x^2 \\
\hline
2x^2-10x+4 & 8x^2-42x+20 \\
 & \pm 8x^2 \mp 40x \pm 16 \\
\hline
 & -2x+4 \\
\hline
\end{array}
$$

For making the given expression a perfect square the remainder must be zero.

Hence

(i) we should add $(2x - 4)$ to the given expression

(ii) we should subtract $(-2x + 4)$ from the given expression

(iii) we should take $-2x + 4 = 0$ to find the value of x. This gives the required value of x i.e., $x = 2$.

EXERCISE 6.3

1. Use factorization to find the square root of the following expressions.
   (i) $4x^2 - 12xy + 9y^2$
   (ii) $x^2 - 1 + \frac{1}{4x^2} \quad (x \neq 0)$
   (iii) $\frac{1}{16}x^2 - \frac{1}{12}xy + \frac{1}{36}y^2$
   (iv) $4(a+b)^2 - 12(a^2 - b^2) + 9(a-b)^2$
   (v) $\frac{4x^6 - 12x^3y^3 + 9y^6}{9x^4 + 24x^2y^2 + 16y^4}$
   (vi) $\left(x + \frac{1}{x}\right)^2 - 4\left(x - \frac{1}{x}\right) \quad (x \neq 0)$
   (vii) $\left(x^2 + \frac{1}{x^2}\right)^2 - 4\left(x + \frac{1}{x}\right)^2 + 12 \quad (x \neq 0)$
   (viii) $(x^2 + 3x + 2)(x^2 + 4x + 3)(x^2 + 5x + 6)$
   (ix) $(x^2 + 8x + 7)(2x^2 - x - 3)(2x^2 + 11x - 21)$
2. Use division method to find the square root of the following expressions.
   (i) $4x^2 + 12xy + 9y^2 + 16x + 24y + 16$
   (ii) $x^4 - 10x^3 + 37x^2 - 60x + 36$
   (iii) $9x^4 - 6x^3 + 7x^2 - 2x + 1$
   (iv) $4 + 25x^2 - 12x - 24x^3 + 16x^4$
   (v) $\frac{x^2}{y^2} - 10\frac{x}{y} + 27 - 10\frac{y}{x} + \frac{y^2}{x^2} \quad (x \neq 0, y \neq 0)$
3. Find the value of k for which the following expressions will become a perfect square.
   (i) $4x^4 - 12x^3 + 37x^2 - 42x + k$ (ii) $x^4 - 4x^3 + 10x^2 - kx + 9$
4. Find the values of l and m for which the following expressions will become perfect squares.
   (i) $x^4 + 4x^3 + 16x^2 + lx + m$ (ii) $49x^4 - 70x^3 + 109x^2 + lx - m$
5. To make the expression $9x^4 - 12x^3 + 22x^2 - 13x + 12$, a perfect square
   (i) what should be added to it?
   (ii) what should be subtracted from it?
   (iii) what should be the value of x?

REVIEW EXERCISE 6

1. **Choose the correct answer.**
2. Find the H.C.F. of the following by factorization.
   $8x^4 - 128,\ 12x^3 - 96$
3. Find the H.C.F. of the following by division method.
   $y^3 + 3y^2 - 3y - 9,\ y^3 + 3y^2 - 8y - 24$
4. Find the L.C.M. of the following by factorization.
   $12x^2 - 75,\ 6x^2 - 13x - 5,\ 4x^2 - 20x + 25$
5. If H.C.F. of $x^4 + 3x^3 + 5x^2 + 26x + 56$ and $x^4 + 2x^3 - 4x^2 - x + 28$ is $x^2 + 5x + 7$, find their L.C.M.
6. Simplify
   (i) $\frac{3}{x^3 + x^2 + x + 1} - \frac{3}{x^3 - x^2 + x - 1}$
   (ii) $\frac{a+b}{a^2 - b^2} \div \frac{a^2 - ab}{a^2 - 2ab + b^2}$
7. Find square root by using factorization
   $\left(x^2 + \frac{1}{x^2}\right) + 10\left(x + \frac{1}{x}\right) + 27 \quad (x \neq 0)$
8. Find square root by using division method.
   $\frac{4x^2}{y^2} + \frac{20x}{y} + 13 - \frac{30y}{x} + \frac{9y^2}{x^2} \quad (x, y \neq 0)$

SUMMARY

- We learned to find the H.C.F. and L.C.M. of algebraic expressions by the methods of factorization and division.
- We established a relation between H.C.F. and L.C.M. of two polynomials $p(x)$ and $q(x)$ given by the formula

$$\textbf{L.C.M.} \times \textbf{H.C.F.} = p(x) \times q(x)$$

  and used it to determine L.C.M. or H.C.F. etc.
- Any unknown expression may be found if three of them are known by using the relation

$$\textbf{L.C.M} \times \textbf{H.C.F} = p(x) \times q(x)$$

- H.C.F. and L.C.M. are used to simplify fractional expressions involving basic operations of $+, -, \times, \div$.
- Determination of square root of algebraic expression by factorization and division methods has been defined and explained.

Animation 7.1: Linear Equations and Inequalities
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Students Learning Outcomes

After studying this unit, the students will be able to:

- Recall linear equation in one variable.
- Solve linear equation with rational coefficients.
- Reduce equations, involving radicals, to simple linear form and find their solutions.
- Define absolute value.
- Solve the equation, involving absolute value, in one variable.
- Define inequalities (>, <) and (> , <)
- Recognize properties of inequalities (i.e., trichotomy, transitive, additive and multiplicative).
- Solve linear inequalities with rational coefficients.

Introduction

In this unit we will extend the study of previously learned skills to the solution of equations with rational coefficients of Unit 2 and the equations involving radicals and absolute value. Finally, after defining inequalities, and recalling their trichotomy, transitive, additive and multiplicative properties we will use them to solve linear inequalities with rational coefficients.

7.1 Linear Equations

7.1.1 Linear Equation

A linear equation in one unknown variable x is an equation of the form

$ax + b = 0$, where $a, b \in$ R and $a \neq 0$

A solution to a linear equation is any replacement or substitution for the variable x that makes the statement true. Two linear equations are said to be equivalent if they have exactly the same solution.

7.1.2 Solving a Linear Equation in One Variable

The process of solving an equation involves finding a sequence of equivalent equations until the variable x is isolated on one side of the equation to give the solution.

Technique for Solving

The procedure for solving linear equations in one variable is summarized in the following box.

- If fractions are present, we multiply each side by the L.C.M. of the denominators to eliminate them.
- To remove parentheses we use the distributive property.
- Combine alike terms, if any, on both sides.
- Use the addition property of equality (add or subtract) to get all the variables on left side and constants on the other side.
- Use the multiplicative property of equality to isolate the variable.
- Verify the answer by replacing the variable in the original equation.

Example 1

Solve the equation $\frac{3x}{2} - \frac{x-2}{3} = \frac{25}{6}$

Solution

Multiplying each side of the given equation by 6, the L.C.M. of denominators 2, 3 and 6 to eliminate fractions, we get

$$9x - 2(x - 2) = 25$$

$$\Rightarrow \quad 9x - 2x + 4 = 25$$

$$\Rightarrow \quad 7x = 21$$

$$\Rightarrow \quad x = 3$$

Check

Substituting x = 3 in original equation,

$$\frac{3}{2}(3) - \frac{3-2}{3} = \frac{25}{6}$$

$$\frac{9}{2} - \frac{1}{3} = \frac{25}{6}$$

$$\frac{25}{6} = \frac{25}{6}$$, Which is true

Since $x = 3$ makes the original statement true, therefore the solution is correct.

Note: Some fractional equations may have no solution.

Example 2

Solve $\frac{3}{y-1} - 2 = \frac{3y}{y-1}, \quad y \neq 1$

Solution

To clear fractions we multiply both sides by the L.C.M. = $y - 1$ and get

$3 - 2(y - 1) = 3y$

$\Rightarrow \quad 3 - 2y + 2 = 3y$

$\Rightarrow \quad -5y = -5$

$\Rightarrow \quad y = 1$

Check

Substituting $y = 1$ in the given equation, we have

$$\frac{3}{1-1} - 2 = \frac{3(1)}{1-1}$$

$$\frac{3}{0} - 2 = \frac{3}{0}$$

But $\frac{3}{0}$ is undefined. So $y = 1$ cannot be a solution.

Thus the given equation has no solution.

Example 3

Solve $\frac{3x-1}{3} - \frac{2x}{x-1} = x, \quad x \neq 1$

Solution

To clear fractions we multiply each side by $3(x - 1)$ with the assumption that

$x - 1 \neq 0$ i.e., $x \neq 1$, and get

$(x - 1)(3x - 1) - 6x = 3x(x - 1)$

$\Rightarrow \quad 3x^2 - 4x + 1 = 3x^2 - 3x$

$\Rightarrow \quad -10x + 1 = -3x$

$\Rightarrow \quad -7x = -1$

$\Rightarrow \quad x = \frac{1}{7}$

Check

On substituting $x = \frac{1}{7}$ the original equation is verified a true statement. That means the restriction $x \neq 1$ has no effect on the solution because $\frac{1}{7} \neq 1$.

Hence our solution $x = \frac{1}{7}$ is correct.

7.1.3 Equations Involving Radicals but Reducible to Linear Form

Redical Equation

When the variable in an equation occurs under a radical, the equation is called a radical equation.

The procedure to solve a radical equation is to eliminate the radical by raising each side to a power equal to the index of the radical.

When raising each side of the equation to a certain power may produce a nonequivalent equation that has more solutions than the original equation. These additional solutions are called extraneous solutions. We must check our answer(s) for such solutions when working with radical equations.

Note: An important point to be noted is that raising each side to an odd power will always give an equivalent equation; whereas raising each side to an even power might not do so.

Example 1

Solve the equations

(a) $\sqrt{2x-3} - 7 = 0$ (b) $\sqrt[3]{3x+5} = \sqrt[3]{x-1}$

Solution

(a) To isolate the radical, we can rewrite the given equation as

$$\sqrt{2x-3} = 7$$

$\Rightarrow$ $2x - 3 = 49$, ... (squaring each side)

$\Rightarrow$ $2x = 52 \Rightarrow x = 26$

Check

Let us substitute $x = 26$ in the original equation. Then

$$\sqrt{2(26)-3} - 7 = 0$$
$$\sqrt{52-3} - 7 = 0$$
$$\sqrt{49} - 7 = 0$$
$$0 = 0$$

Hence the solution set is {26}.

(b) We have

$\sqrt[3]{3x+5} = \sqrt[3]{x-1}$ ..... (given)

$\Rightarrow$ $3x + 5 = x - 1$, ... (taking cube of each side)

$\Rightarrow$ $2x = -6 \Rightarrow x = -3$

Check

We substitute $x = -3$ in the original equation. Then

$$\sqrt[3]{3(-3)+5} = \sqrt[3]{-3-1} \Rightarrow \sqrt[3]{-4} = \sqrt[3]{-4}$$

Thus $x = -3$ satisfies the original equation.

Here $\sqrt[3]{-4}$ is a real number because we raised each side of the equation to an odd power.

Thus the solution set = {–3}

Example 2

Solve and check:

$$\sqrt{5x-7} - \sqrt{x+10} = 0$$

Solution

When two terms of a radical equation contain variables in the radicand, we express the equation such that only one of these terms is on each side. So we rewrite the equation in this form to get

$$\sqrt{5x-7} = \sqrt{x+10}$$

$5x - 7 = x + 10$, ... (squaring each side)

$4x = 17 \Rightarrow x = \frac{17}{4}$

Check

Substituting $x = \frac{17}{4}$ in original equation.

$$\sqrt{5x-7} - \sqrt{x+10} = 0$$
$$\sqrt{5\left(\frac{17}{4}\right)-7} - \sqrt{\frac{17}{4}+10} = 0$$
$$\sqrt{\frac{57}{4}} - \sqrt{\frac{57}{4}} = 0$$
$$0 = 0$$

i.e., $x = \frac{17}{4}$ makes the given equation a true statement.

Thus solution set = $\left\{\frac{17}{4}\right\}$.

Example 3

Solve $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

Solution

$$\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$$

Squaring both sides we get Squaring both sides we get

$$x + 7 + x + 2 + 2\sqrt{(x+7)(x+2)} = 6x + 13$$

$\Rightarrow$ $2\sqrt{x^2+9x+14} = 4x + 4$

$\Rightarrow$ $\sqrt{x^2+9x+14} = 2x + 2$

Squaring again

$x^2 + 9x + 14 = 4x^2 + 8x + 4$

$\Rightarrow \quad 3x^2 - x - 10 = 0$

$\Rightarrow \quad 3x^2 - 6x + 5x - 10 = 0$

$\Rightarrow \quad 3x(x - 2) + 5(x - 2) = 0$

$\Rightarrow \quad (x - 2)(3x + 5) = 0$

$\Rightarrow \quad x = 2, -\frac{5}{3}$

On checking, we see that $x = 2$ satisfies the equation, but $x = -\frac{5}{3}$ does not satisfy the equation. So solution set is {2} and $x = -\frac{5}{3}$ is an extraneous root.

EXERCISE 7.1

1. Solve the following equations.

(i) $\frac{2}{3}x - \frac{1}{2}x = x + \frac{1}{6}$ (ii) $\frac{x-3}{3} - \frac{x-2}{2} = -1$

(iii) $\frac{1}{2}\left(x - \frac{1}{6}\right) + \frac{2}{3} = \frac{5}{6} + \frac{1}{3}\left(\frac{1}{2} - 3x\right)$ (iv) $x + \frac{1}{3} = 2\left(x - \frac{2}{3}\right) - 6x$

(v) $\frac{5(x-3)}{6} - x = 1 - \frac{x}{9}$ (vi) $\frac{x}{3x-6} = 2 - \frac{2x}{x-2}, \; x \neq 2$

(vii) $\frac{2x}{2x+5} = \frac{2}{3} - \frac{5}{4x+10}, \; x \neq -\frac{5}{2}$ (viii) $\frac{2x}{x-1} + \frac{1}{3} = \frac{5}{6} + \frac{2}{x-1}, \; x \neq 1$

(ix) $\frac{2}{x^2-1} - \frac{1}{x+1} = \frac{1}{x+1}, \; x \neq \pm 1$ (x) $\frac{2}{3x+6} = \frac{1}{6} - \frac{1}{2x+4}, \; x \neq -2$

2. Solve each equation and check for extraneous solution, if any.

(i) $\sqrt{3x+4} = 2$ (ii) $\sqrt[3]{2x-4} - 2 = 0$

(iii) $\sqrt{x-3} - 7 = 0$ (iv) $2\sqrt{t+4} = 5$

(v) $\sqrt[3]{2x+3} = \sqrt[3]{x-2}$ (vi) $\sqrt[3]{2-t} = \sqrt[3]{2t-28}$

(vii) $\sqrt{2t+6} - \sqrt{2t-5} = 0$ (viii) $\sqrt{\frac{x+1}{2x+5}} = 2, \; x \neq -\frac{5}{2}$

7.2 Equation Involving Absolute Value

Another type of linear equation is the one that contains absolute value. To solve equations involving absolute value we first give the following definition.

7.2.1 Absolute Value

The absolute value of a real number 'a' denoted by $|a|$, is defined a

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

e.g., $|6| = 6$, $|0| = 0$ and $|-6| = -(-6) = 6$.

Some properties of Absolute Value

If $a, b \in R$, then

(i) $|a| \geq 0$ (ii) $|-a| = |a|$

(iii) $|ab| = |a| . |b|$ (iv) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, b ≠ 0

7.2.2 Solving Linear Equations Involving Absolute Value

Keeping in mind the definition of absolute value, we can immediately say that

$|x| = 3$ is equivalent to $x = 3$ or $x = -3$,

because $x = +3$ or $x = -3$ make $|x| = 3$ a true statement.

For solving an equation involving absolute value, we express the given equation as an equivalent compound sentence and solve each part separately.

Example 1

Solve and check, $|2x + 3| = 11$

Solution

By definition, depending on whether $(2x + 3)$ is positive or

negative, the given equation is equivalent to

$$+(2x + 3) = 11 \quad \text{or} \quad -(2x + 3) = 11$$

In practice, these two equations are usually written as

$$2x + 3 = +11 \quad \text{or} \quad 2x + 3 = -11$$
$$2x = 8 \quad \text{or} \quad 2x = -14$$
$$x = 4 \quad \text{or} \quad x = -7$$

Check

Substituting $x = 4$, in the original equation, we get

$$|2(4) + 3| = 11$$

i.e., $11 = 11$, true

New substituting $x = -7$, we have

$$|2(-7) + 3| = 11$$
$$|-11| = 11$$
$$11 = 11, \text{ true}$$

Hence $x = 4, -7$ are the solutions to the given equation.

or Solution set = $\{-7, 4\}$

Note: For an equation like $3|x - 1| - 6 = 8$, do not forget to isolate the absolute value expression on one side of the equation before writing the equivalent equations. In the equation under consideration we must first write it as

$$|x - 1| = 14/3$$

Example 2

Solve $|8x - 3| = |4x + 5|$

Solution

Since two numbers having the same absolute value are either equal or differ in sign, therefore, the given equation is equivalent to

$$8x - 3 = 4x + 5 \quad \text{or} \quad 8x - 3 = -(4x + 5)$$
$$4x = 8 \quad \text{or} \quad 12x = -2$$
$$x = 2 \quad \text{or} \quad x = -1/6$$

On checking we find that $x = 2$, $x = -\frac{1}{6}$ both satisfy the original equation.

Hence the solution set $\{-\frac{1}{6}, 2\}$.

Sometimes it may happen that the solution(s) obtained do not satisfy the original equation. Such solution(s) (called extraneous) must be rejected. Therefore, it is always advisable to check the solutions in the original equation.

Example 3

Solve and check $|3x + 10| = 5x + 6$

Solution

The given equation is equivalent to

$$\pm(3x + 10) = 5x + 6$$

i.e., $3x + 10 = 5x + 6$ or $3x + 10 = -(5x + 6)$

$$-2x = -4 \quad \text{or} \quad 8x = -16$$
$$x = 2 \quad \text{or} \quad x = -2$$

On checking in the original equation we see that $x = -2$ does not satisfy it. Hence the only solution is $x = 2$.

EXERCISE 7.2

1. Identify the following statements as True or False.

(i) $|x| = 0$ has only one solution. ......

(ii) All absolute value equations have two solutions. ......

(iii) The equation $|x| = 2$ is equivalent to $x = 2$ or $x = -2$. ......

(iv) The equation $|x - 4| = -4$ has no solution. ......

(v) The equation $|2x - 3| = 5$ is equivalent to $2x - 3 = 5$ or $2x + 3 = 5$. ......

2. Solve for x

(i) $|3x - 5| = 4$

(ii) $\frac{1}{2}|3x + 2| - 4 = 11$

(iii) $|2x + 5| = 11$

(iv) $|3 + 2x| = |6x - 7|$

(v) $|x + 2| - 3 = 5 - |x + 2|$

(vi) $\frac{1}{2}|x + 3| + 21 = 9$

(vii) $\left|\frac{3x - 5}{2}\right| - \frac{1}{3} = \frac{2}{3}$

(viii) $\left|\frac{x + 5}{2 - x}\right| = 6$

7.3 Linear Inequalities

In Unit 2 ,we discussed an important comparing property of ordering real numbers. This order relation helps us to compare two real numbers 'a' and 'b' when $a \neq b$. This comparability is of primary importance in many applications. We may compare prices, heights, weights, temperatures, distances, costs of manufacturing, distances, time etc. The inequality symbols < and > were introduced by an English mathematician Thomas Harriot (1560 — 1621).

7.3.1 Defining Inequalities

Let a, b be real numbers. Then a is greater than b if the difference $a - b$ is positive and we denote this order relation by the inequality $a > b$. An equivalent statement is that in which b is less than a, symbolised by $b < a$ Similarly, if $a - b$ is negative, then a is less than b and expressed in symbols as $a < b$.

Sometimes we know that one number is either less than another number or equal to it. But we do not know which one is the case. In such a situation we use the symbol "<" which is read as "less than or equal to". Likewise, the symbol "≥" is used to mean "greater than or equal to". The symbols < , >, and > are also called inequality signs. The inequalities $x > y$ and $x < y$ are known as strict (or strong) whereas the inequalities where as $x \leq y$ and $y \leq x$ are called non-strict (or weak).

If we combine $a < b$ and $b < c$ we get a double inequality written in a compact form as $a < b < c$ which means "b lies between a and c" and read as "a is less than b less than c" Similarly, "$a \leq b \leq c$" is read as "b is between a and c, inclusive."

A linear inequality in one variable x is an inequality in which the variable x occurs only to the first power and has the standard form

$$ax + b < 0, \; a \neq 0$$

where a and b are real numbers. We may replace the symbol < by >, ≤ or ≥ also.

7.3.2 Properties of Inequalitie

The properties of inequalities which we are going to use in solving linear inequalities in one variable are as under.

1 Law of Trichotomy

For any $a, b \in$ R, one and only one of the following statements is true.

$$a < b \quad \text{or} \quad a = b, \quad \text{or} \quad a > b$$

An important special case of this property is the case for $b = 0$; namely,

$$a < 0 \text{ or } a = 0 \text{ or } a > 0 \text{ for any } a \in \text{R}.$$

2 Transitive Property

Let $a, b, c \in$ R.

(i) If $a > b$ and $b > c$, then $a > c$

(ii) If $a < b$ and $b < c$, then $a < c$

3 Additive Closure Property For a,b,c ∈ **R,**

(i) If $a > b$, then $a + c > b + c$

If $a < b$, then $a + c < b + c$

(ii) If $a > 0$ and $b > 0$, then $a + b > 0$

If $a < 0$ and $b < 0$, then $a + b < 0$

4 Multiplicative Property

Let a, b, c, d ∈ R

(i) If $a > 0$ and $b > 0$, then $ab > 0$, whereas $a < 0$ and $b < 0 \Rightarrow ab > 0$

(ii) If $a > b$ and $c > 0$, then $ac > bc$

or if $a < b$ and $c > 0$, then $ac < bc$

(iii) If $a > b$ and $c < 0$, then $ac < bc$

or if $a < b$ and $c < 0$, then $ac > bc$

The above property (iii) states that the sign of inequality is reversed

(iv) If $a > b$ and $c > d$, then $ac > bd$

7.4. Solving Linear Inequalities

The method of solving an algebraic inequality in one variable is explained with the help of following examples.

Example 1

Solve $9 - 7x > 19 - 2x$, where $x \in R$.

Solution

$9 - 7x > 19 - 2x$

$9 - 5x > 19$ ...... (Adding $2x$ to each side)

$-5x > 10$ ...... (Adding -9 to each side)

$x < -2$ ...... (Multiplying each side by $-\frac{1}{5}$)

Hence the solution set = $\{x \mid x < -2\}$

Example 2

Solve $\frac{1}{2}x - \frac{2}{3} \le x + \frac{1}{3}$, where $x \in R$.

Solution

$\frac{1}{2}x - \frac{2}{3} \le x + \frac{1}{3}$,

To clear fractions we multiply each side by 6, the L.C.M. of 2 and 3 and get

$6\left[\frac{1}{2}x - \frac{2}{3}\right] \le 6\left[x + \frac{1}{3}\right]$

or $3x - 4 \le 6x + 2$

or $3x \le 6x + 6$

or $-3x \le 6$

or $x \ge -2$

Hence the solution set = $\{x \mid x \ge -2\}$.

Example 3

Solve the double inequality $< \frac{1-2x}{3} < 1$, where $x \in R$.

Solution

The given inequality is a double inequality and represents two separate inequalities

$$-2 < \frac{1-2x}{3} \quad \text{and} \quad \frac{1-2x}{3} < 1$$

$$-2 < \frac{1-2x}{3} < 1$$

or $-6 < 1 - 2x < 3$

or $-7 < -2x < 2$

or $\frac{7}{2} > x > -1$

i.e., $-1 < x < 3.5$

So the solution set is $\{x \mid -1 < x < 3.5\}$.

Example 4

Solve the inequality $4x - 1 \le 3 \le 7 + 2x$, where $x \in R$.

Solution

The given inequality holds if and only if both the separate inequalities $4x - 1 \le 3$ and $3 \le 7 + 2x$ hold. We solve each of these inequalities separately.

The first inequality $4x - 1 \le 3$

gives $4x \le 4$ i.e., $x \le 1$ ...... (i)

and the second inequality $3 \le 7 + 2x$ yields $-4 \le 2x$

i.e., $-2 \le x$ which implies $x \ge -2$ ...... (ii)

Combining (i) and (ii), we have $-2 \le x \le 1$

Thus the solution set = $\{x \mid -2 \le x \le 1\}$.

EXERCISE 7.3

1. Solve the following inequalities

(i) $3x + 1 < 5x - 4$

(ii) $4x - 10.3 \le 21x - 1.8$

(iii) $4 - \frac{1}{2}x \ge -7 + \frac{1}{4}x$

(iv) $x - 2(5 - 2x) \ge 6x - 3\frac{1}{2}$

(v) $\frac{3x+2}{9} - \frac{2x+1}{3} > -1$

(vi) $3(2x + 1) - 2(2x + 5) < 5(3x - 2)$

(vii) $3(x - 1) - (x - 2) > -2(x + 4)$ (viii) $2\frac{2}{3}x + \frac{2}{3}(5x - 4) > -\frac{1}{3}(8x + 7)$

2. Solve the following inequalities

(i) $-4 < 3x + 5 < 8$ (ii) $-5 \le \frac{4 - 3x}{2} < 1$

(iii) $-6 < \frac{x - 2}{4} < 6$ (iv) $3 \ge \frac{7 - x}{2} \ge 1$

(v) $3x - 10 \le 5 < x + 3$ (vi) $-3 \le \frac{x - 4}{-5} < 4$

(vii) $1 - 2x < 5 - x \le 25 - 6x$ (viii) $3x - 2 < 2x + 1 < 4x + 17$

REVIEW EXERCISE 7

1. **Choose the correct answer.**

2. Identify the following statements as True or False

(i) The equation $3x - 5 = 7 - x$ is a linear equation. ......

(ii) The equation $x - 0.3x = 0.7x$ is an identity. ......

(iii) The equation $-2x + 3 = 8$ is equivalent to $-2x = 11$. ......

(iv) To eliminate fractions, we multiply each side of an equation by the L.C.M.of denominators.......

(v) $4(x + 3) = x + 3$ is a conditional equation. ......

(vi) The equation $2(3x + 5) = 6x + 12$ is an inconsistent equation.....

(vii) To solve $\frac{2}{3}x = 12$ we should multiply each side by $\frac{2}{3}$ ....

(viii) Equations having exactly the same solution are called equivalent equations. ......

(ix) A solution that does not satisfy the original equation is called extraneous solution. ......

3. Answer the following short questions.

(i) Define a linear inequality in one variable.

(ii) State the trichotomy and transitive properties of inequalities.

(iii) The formula relating degrees Fahrenheit to degrees Celsius is $F = \frac{9}{5}C + 32$. For what value of C is F < 0?

(iv) Seven times the sum of an integer and 12 is at least 50 and at most 60. Write and solve the inequality that expresses this relationship.

4. Solve each of the following and check for extraneous solution if any

(i) $\sqrt{2t + 4} = \sqrt{t - 1}$ (ii) $\sqrt{3x - 1} - 2\sqrt{8 - 2x} = 0$

5. Solve for x

(i) $|3x + 14| - 2 = 5x$ (ii) $\frac{1}{3}|x - 3| = \frac{1}{2}|x + 2|$

6. Solve the following inequality

(i) $-\frac{1}{3}x + 5 \le 1$ (ii) $-3 < \frac{1 - 2x}{5} < 1$

SUMMARY

- Linear Equation in one variable x is $ax + b = 0$ where $a, b \in R, a \ne 0$.
- Solution to the equation is that value of x which makes it a true statement.
- An inconsistent equation is that whose solution set is ϕ.
- Additive property of equality:
  If $a = b$, then $a + c = b + c$
  and $a - c = b - c$. $\forall\ a, b, c \in R$
- Multiplicative property of equality: If $a = b$, then $ac = bc$
- Cancellation property:
  If $a + c = b + c$, then $a = b$
  If $ac = bc, c \ne 0$ then $a = b, \forall\ a, b, c \in R$
- To solve an equation we find a sequence of equivalent equations to isolate the variable x on one side of the equality to get solution.

- A radical equation is that in which the variable occurs under the radical. It must be checked for any extraneous solution(s)
- Absolute value of a real number a is defined as

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

- Properties of Absolute value:

  if $a, b \in$ R, then

  (i) $|a| \geq 0$

  (ii) $|-a| = |a|$

  (iii) $|ab| = |a|.|b|$

  (iv) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ $b \neq 0$

  (v) $|x| = a$ is equivalent to $x = a$ or $x = -a$

- Inequality symbols are $<, >, \leq, \geq$
- A linear inequality in one variable x is $ax + b < 0, a \neq 0$
- Properties of Inequality:

  (a) Law of Trichotomy

  If $a, b \in$ R then $a < b$ or $a = b$ or $a > b$

  (b) Transitive laws

  If $a > b$ and $b > c$, then $a > c$

  (c) Multiplication and division:

  (i) If $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$

  (ii) If $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$

version: 1.1

CHAPTER

8

LINEAR GRAPHS & THEIR APPLICATION

Animation 8.1: Linear Graphs
Source & Credit: eLearn.punjab

Students Learning Outcomes

After studying this unit, the students will be able to:

- Identify pair of real numbers as an ordered pair.
- Recognize an ordered pair through different examples.
- Describe rectangular or Cartesian plane consisting of two number lines interesting at right angles at the point O.
- Identify origin (***O***) and coordinate axes (horizontal and vertical axes or x-axis and y-axis) in the rectangular plane.
- Locate an ordered pair (a, b) as a point in the rectangular plane and recognize.
  - a as the x-coordinate (or abscissa),
  - b as the y-coordinate (or ordinate).
- Draw different geometrical shapes (e.g., line segment, triangle and rectangle etc.) by joining a set of given points.
- Construct a table for pairs of values satisfying a linear equation in two variables.
- Plot the pairs of points to obtain the graph of a given expression.
- Choose an appropriate scale to draw a graph.
- Draw a graph of
  - an equation of the form $y = c$,
  - an equation of the form $x = a$,
  - an equation of the form $y = mx$,
  - an equation of the form $y = mx + c$.
- Draw a graph from a given table of (discrete) values.
- Solve appropriate real life problems.
- Interpret conversion graph as a linear graph relating to two quantities which are in direct proportion.
- Read a given graph to know one quantity corresponding to another.
- Read the graph for conversions of the form.
  - miles and kilometers, acres and hectares,
  - degrees Celsius and degrees Fahrenheit,
  - Pakistani currency and another currency, etc.
- Solve simultaneous linear equations in two variables using graphical method.

8.1 Cartesian Plane and Linear Graphs

8.1.1 An Ordered Pair of Real Numbers

An ordered pair of real numbers x and y is a pair (x, y) in which elements are written in specific order.

i.e., (i) (x, y) is an ordered pair in which first element is x and second is y. such that $(x, y) \neq (y, x)$ where, $x \neq y$.

(ii) (2, 3) and (3, 2) are two different ordered pairs.

(iii) $(x, y) = (m, n)$ only if $x = m$ and $y = n$.

8.1.2 Recognizing an Ordered Pair

In the class room the seats of a student is the example of an ordered pair. For example, the seat of the student A is at the 5th place in the 3rd row, so it corresponds to the ordered pair (3, 5). Here 3 shows the number of the row and 5 shows its seat number in this row.

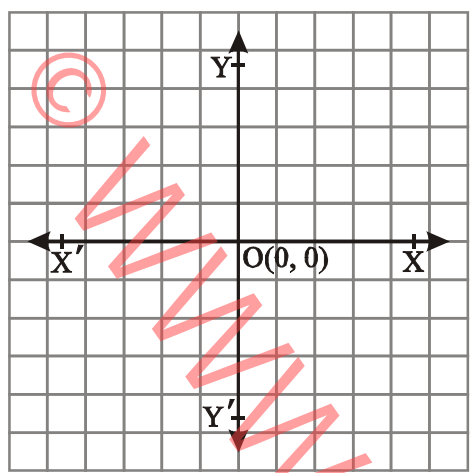
Similarly an ordered pair (4, 3) represents a seat located to a student A in the examination hall is at the 4th row and 3rd column i.e. 3rd place in the 4th row.

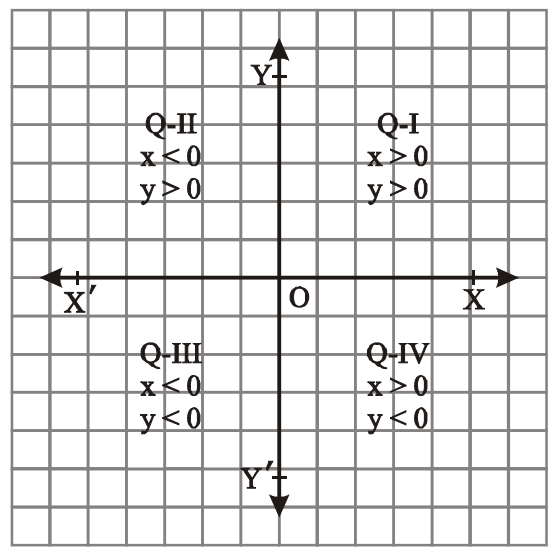
8.1.3 Cartesian Plane

The cartesian plane establishes one-to-one correspondence between the set of ordered pairs $R \times R = \{(x, y) \mid x, y \in R\}$ and the points of the Cartesian plane.

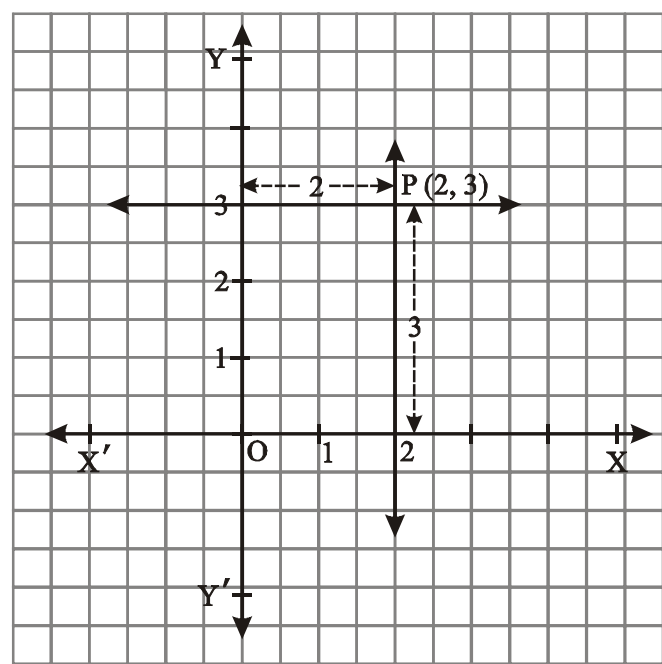
In plane two mutually perpendicular straight lines are drawn. The lines are called the coordinate axes. The point **O**, where the two lines meet is called origin. This plane is called the coordinate plane or the Cartesian plane.

8.1.4 Identification of Origin and Coordiante Axes

The horizontal line **XOX′** is called the x-axis and the vertical line **YOY′** is called the y-axis. The point **O** where the x-axis and y-axis meet is called the origin and it is denoted by O(0, 0).

We have noted that each point in the plane either lies on the axes of the coordinate plane or in any one of quadrants of the plane namely **XOY, YOX′, X′OY′** and **Y′OX** called the first, sceond, thirdand the fourth quadrants of the planesubdivided by the coordinate axes of the plane. They are denoted by **Q-I, Q-II, Q-III** and **Q-IV** respectively.

The signs of the coordinates of the points **(*x, y*)** are shown below;

e.g., 1. The point (–3, –1) lies in Q-III. 2. The point (2, –3) lies in Q-IV.
3. The point (2, 5) lies in Q-I. 4. The point (2, 0) lies on *x*-axis.

8.1.5 Location of the Point P(a, b) in the Plane Corresponding to the Ordered Pair (a,b)

Let (*a, b*) be an ordered pair of R x R.

In the reference system, the real number a is measured along x-axis, ***OA* = *a*** units away from the origin along ***OX*** (if **a** > 0) and the real number b along y-axis, ***OB* = *b*** units away from the origin along **OY** (if **b** > 0). From ***B*** on ***OY***, draw the line parallel to x-axis and from A on OX draw line parallel to y-axis. Both the lines meet at the point P. Then the point P corresponds to the ordered pair **(*a, b*)**.

In the graph shown above 2 is the x-coordinate and 3 is the y-coordinate of the point **P** which is denoted by **P(2, 3).**

In this way coordinates of each point in the plane are obtained. The x-coordinate of the point is called abscissa of the point ***P(x, y)*** and the y-coordinate is called its ordinate.

1. Each point P of the plane can be identified by the coordinates of the pair **(*x, y*)** and is represented by **P(*x*, y).**
2. All the points of the plane have y-coordinate, ***y*** = 0 if they lie on the x-axis. i.e., P(-2, 0) lies on the axis.
3. All the points of the plane have x-coordinate ***x*** = 0 if they lie on the y-axis, i.e., Q(0, 3) lies on the y-axis.

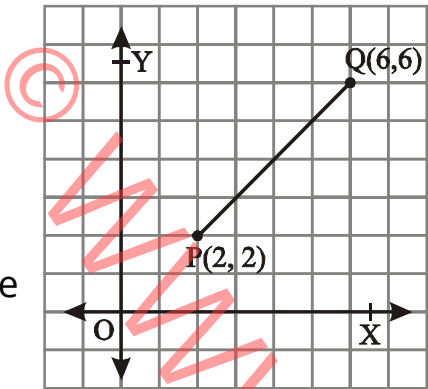
8.1.6 Drawing different Geometrical Shapes of Cartesian Plane

We define first the idea of collinear points before going to form geometrical shapes.

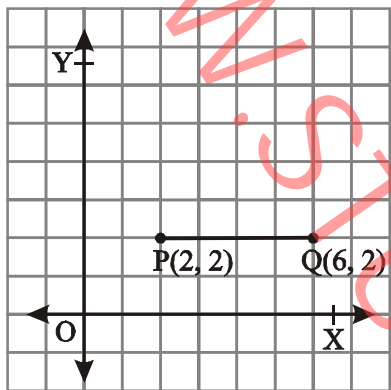
(a) Line-Segment

Example 1:

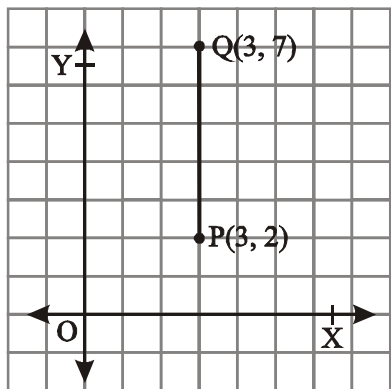
Let P(2, 2) and Q(6, 6) are two points.

1. Plot points P and Q.
2. Join the points P and Q, we get the line segment PQ. It is represented by $\overline{\mathbf{PQ}}$.

Example 2:

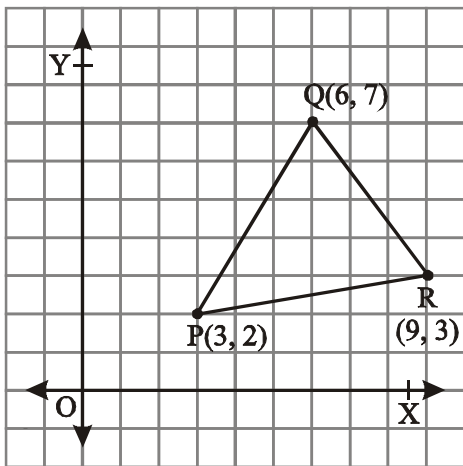
Plot points P(2, 2) and Q(6, 2). By joining them, we get a line segment PQ parallel to x-axis.

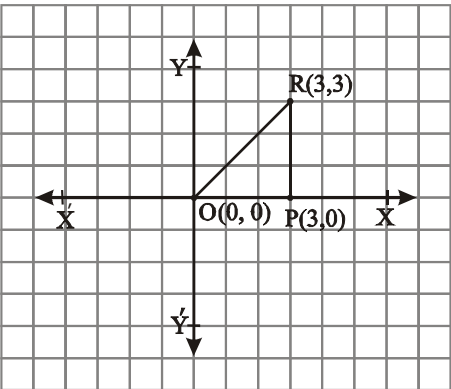
Where ordinate of both points is equal.

Example 3:

Plot points B(3, 2) and Q(3, 7). By joining them, we get a line segment PQ parallel to y-axis.

In this graph abcissas of both the points are equal.

(b) Triangle

Example 1:

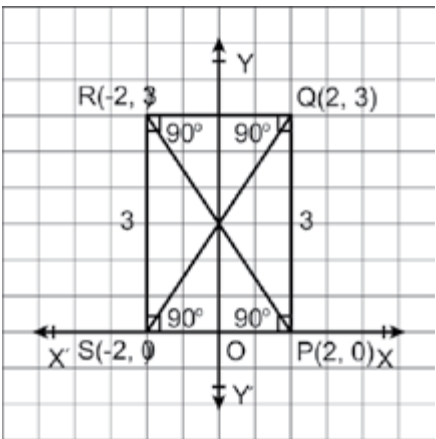
Plot the points P(3, 2), Q(6, 7) and R(9, 3). By joining them, we get a triangle PQR.

Example 2:

For points O(0, 0), P(3, 0) and R(3, 3), the triangle OPR is constructed as shown by the side.

(c) Rectangle

Example:

Plot the points P(2, 0), Q(2, 3), S(–2, 0) and R(–2, 3). Joining the points P, Q, R and S, we get a rectangle PQRS.

Along y-axis,

2 (length of square) = 1

8.1.7 Construction of a Table for Pairs of Values Satisfying a Linear Equation in Two Variables.

Let $2x + y = 1$ (i)

be a linear equation in two variables x and y.

The ordered pair (x, y) satisfies the equation and by varying x, corresponding y is obtained.

We express (i) in the forms

$\boxed{y = -2x + 1}$ (ii)

The pairs (x, y) which satisfy (ii) are tabulated below.

x	y	(x, y)
–1	3	(–1, 3)
0	1	(0, 1)
1	–1	(1, –1)
3	–5	(3, –5)

at $x = -1$, $y = (-2)(-1) + 1 = 2 + 1 = 3$

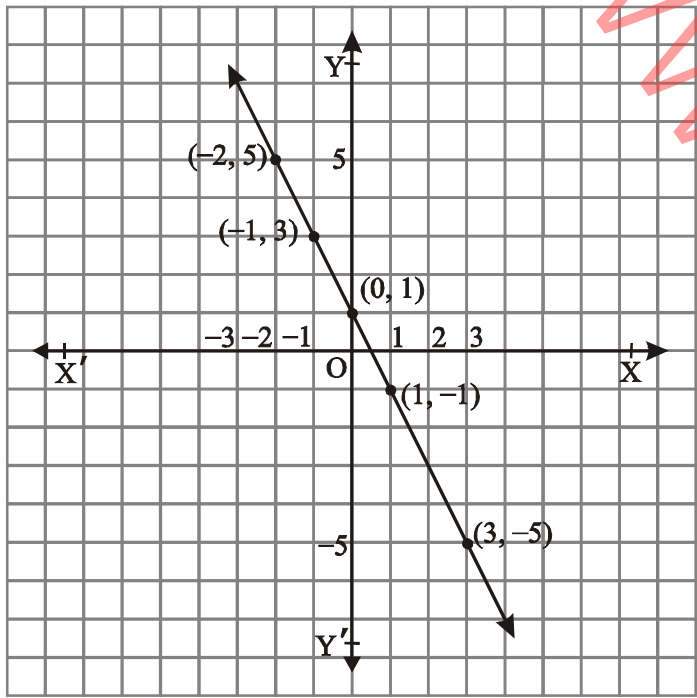
at $x = 0$, $y = (-2)(0) + 1 = 0 + 1 = 1$

at $x = 1$, $y = (-2)(1) + 1 = -2 + 1 = -1$

at $x = 3$, $y = -2(3) + 1 = -5$

Similarly all the points can be computed, the ordered pairs of which do satisfy the equation (i).

8.1.8 Plotting the points to get the graph

Now we plot the points obtained in the table. Joining these points we get the graph of the equation. The graph of **y** = –2*x* + 1 is shown on the next page.

8.1.9 Scale of Graph

To draw the graph of an equation we choose a scale e.g. 1 small square represents 2 meters or 1 small square length represents 10 or 5 meters. It is selected by keeping in mind the size of the paper. Some times the same scale is used for both *x* and **y** coordinates and some times we use different scales for *x* and y-coordinate depending on the values of the coordinates.

8.1.10 Drawing Graphs of the following Equations

(a) **y** = **c**, where **c** is constant.
(b) *x* = *a*, where *a* is constant.
(c) **y** = **m***x*, where **m** is constant.
(d) **y** = **m***x* + **c**, where **m** and **c** both are constants.

By drawing the graph of an equation is meant to plot those points in the plane, which form the graph of the equation (by joining the plotted points).

(a) The equation **y** = **c** is formed in the plane by the set,

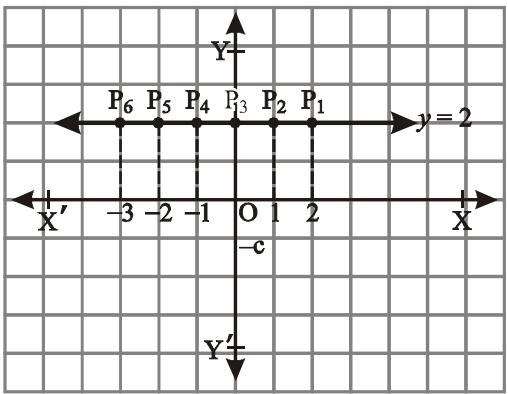
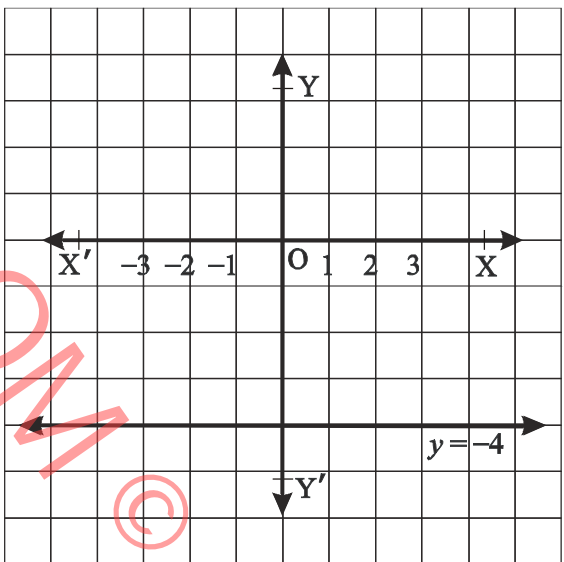
S = {(*x*, **c**): *x* lies on the x-axis} sub set RxR.

The procedure is explained with the help of following examples.

Consider the equation **y** = 2 The set S is tabulated as;

The set S is tabulated as;

x	....	-3	-2	-1	0	1	2	....
y	2	2	2	2	2	2	2	2

The points of S are plotted in the plane.

Similarly graph of **y** = –4 is shown as:

So, the graph of the equation of the type **y** = **c** is obtained as:

(i) the straight line
(ii) the line is parallel to x-axis

(iii) the line is above the x-axis at a distance **c** units if **c** > 0

(iv) the line (shown as y = –4) is below the x-axis at the distance c units as **c** < 0

(v) the line is that of x-axis at the distance **c** units if **c** = 0

(b) The equation, $\boldsymbol{x} = \boldsymbol{a}$ is drawn in the plane by the points of the set

S = {(a, y): y ∈ R}

The points of S are tabulated as follows:

x	a	a	a	a	a	a	a	a	...
y	...	–2	–1	0	1	2	3	4	...

The points of S are plotted in the plane as, ... (a, –2), (a, –1), (a, 0), (a, 1), (a, 2), ... etc.

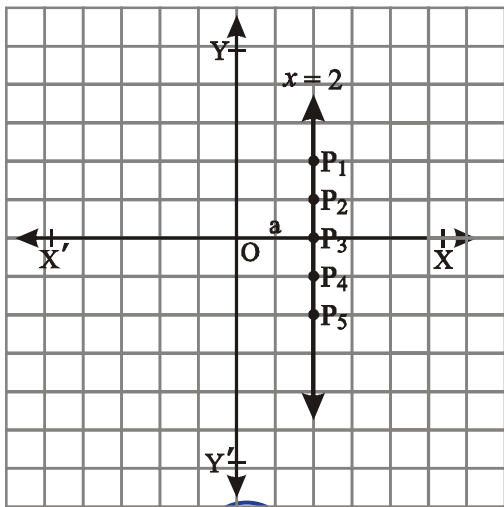
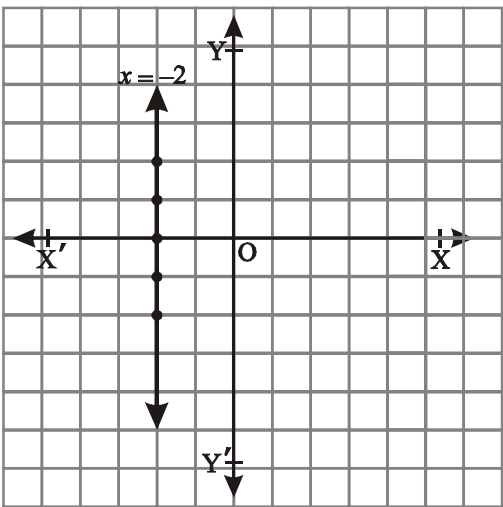
The point (a, 0) on the graph of the equation $x = a$ lies on the x-axis while (a, y) is above the x-axis if y > 0 and below the x-axis if y < 0. By joining the points, we get the line.

The procedure is explained with the help of following examples.

Consider the equation $x = 2$

Table for the points of equation is as under

x	2	2	2	2	2	2	... 2 ...
y	...	-2	-1	0	1	2	...

Thus, graph of the equation **x** = 2 is shown as:

Similarly graph for equation $\boldsymbol{x} = -2$ is shown as:

So, the graph of the equation of the type $x = a$ is obtained as:

(i) the straight line

(ii) the line parallel to the y-axis

(iii) the line is on the right side of y-axis at distance "a" units if a > 0.

(iv) the line $x = -2$ is on the left side of y-axis at the distance a units as a < 0.

(v) the line is y-axis if a = 0.

(c) The equation $y = mx$, (for a fixed $m \in$ R) is formed by the points of the set W = {(x, mx) : $x \in$ R}

i.e. W = {....., (–2, –2**m**), (–1, –**m**), (0, 0), (1, **m**), (2, 2**m**), ... }.

The points corresponding to the ordered pairs of the set W are tabulated below:

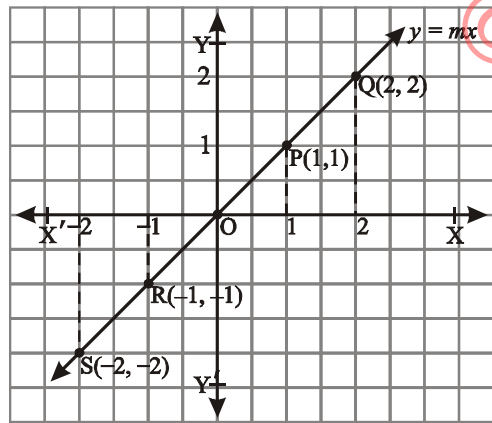
$\boldsymbol{x}$	....	–2	–1	0	1	2	....
$\boldsymbol{y}$	....	–2**m**	–**m**	0	**m**	2**m**	....

The procedure is explained with the help of following examples.

Consider the equation $y = x$, where m = 1

Table of points for equation is as under:

$\boldsymbol{x}$	...	–2	–1	0	1	2	...
$\boldsymbol{y}$	...	–2	–1	0	1	2	...

The points are plotted in the plane as follows:

By joining the plotted points the graph of the equation of the type **y = mx** is,

(i) the straight line

(ii) it passes through the origin O(0, 0)

(iii) **m** is the slope of the line

(iv) the graph of line splits the plane into two equal parts. If **m** = 1, then the line becomes the graph of the equation **y = x**.

(v) If m = -1 then line is the graph of the equation $y = -x$.

(vi) the line meets both the axes at the origin and no other poin

(d) Now we move to a generalized form of the equation, i.e.,

$y = mx + c$, where $m, c \neq 0$.

The points corresponding to the ordered pairs of the

$S = \{(x, mx + c): m, c\ (\neq 0) \in R\}$ are tabulated below

x	0	1	2	3	.....	x	.....
y	c	m + c	2m + c	3m + c	.....	mx + c	.....

The procedure is explained with the help of following examples.

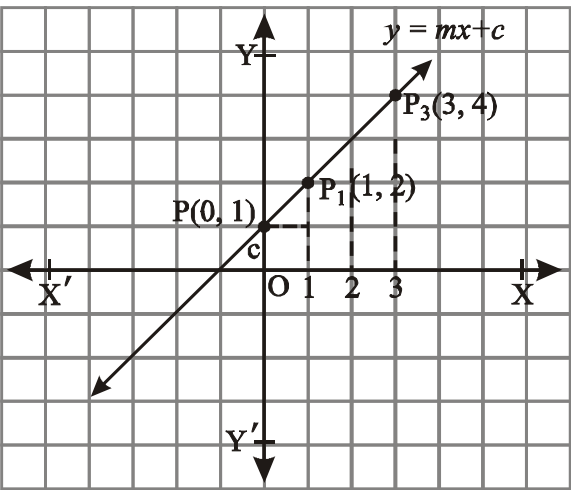
Consider the equation

$y = x + 1$, where m = 1, c = 1

We get the table

x	... 0	1	2	3	...	...	...
y	... 1	2	3	4	...	...	...

8. Linear Graphs & their Application eLearn.Punjab

These points are plotted in plane as below:

We see that

(i) **y = mx + c** represents the graph of a line.

(ii) It does not pass through the origin O(0, 0).

(iii) It has intercept **c** units along the y-axis away from the origin.

(iv) **m** is the slope of the line whose equation is **y = mx + c.**

In particular if

(i) **c** = 0, then **y = mx** passes through the origin.

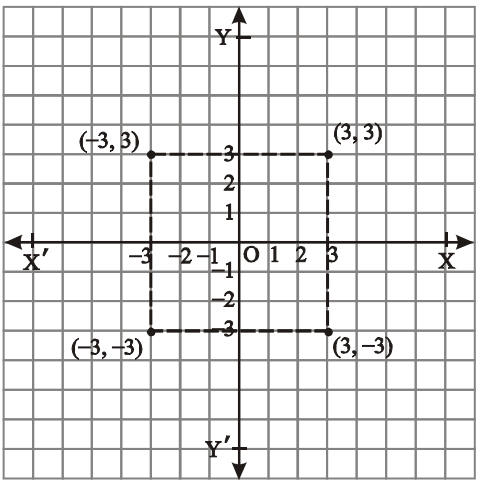
(ii) **m** = 0, then the line **y = c** is parallel to x-axis.

8.1.11 Drawing Graph from a given Table of Discrete Values

If the points are discrete the graph is just the set of points. The points are not joined.

For example, the following table of discrete values is plotted as:

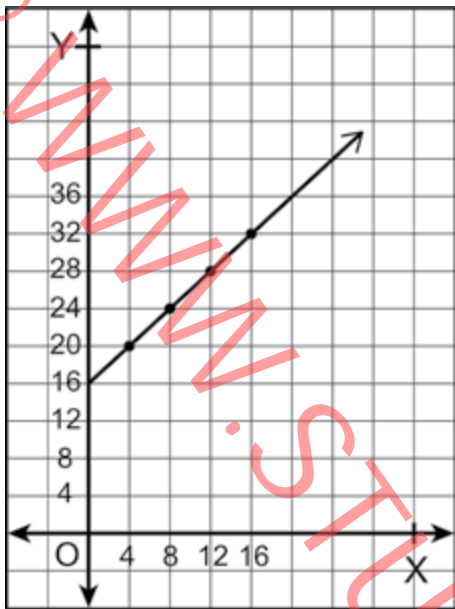
x	3	3	-3	-3
y	3	-3	3	-3

So, the dotted square shows the graph of discrete values.

8.1.12 Solving Real Life Problems

We often use the graph to solve the real life problems. With the help of graph, we can determine the relation or trend between the both quantities.

We learn the procedure of drawing graph of real life problems with the help of following examples.

Example:

Equation $\mathbf{y} = x + 16$ shows the relationship between the age of two person i.e. if the age of one person is x, then the age of other person is **y**. Draw the graph.

Solution

We know that $\mathbf{y} = x + 16$

Table of points for equation is given as:

x	0	4	8	12	16	...
y	16	20	24	28	32	...

By plotting the points we get the graph of a straight line as shown in the figure.

EXERCISE 8.1

1. Determine the quadrant of the coordinate plane in which the following points lie: P(–4, 3), Q(–5, –2), R(2, 2) and S(2, –6).
2. Draw the graph of each of the following

(i) $x = 2$ (ii) $x = -3$ (iii) $\mathbf{y} = -1$

(iv) $\mathbf{y} = 3$ (v) $\mathbf{y} = 0$ (vi) $x = 0$

(vii) $\mathbf{y} = 3x$ (viii) $-\mathbf{y} = 2x$ (ix) $\frac{1}{2} = x$

(x) $3\mathbf{y} = 5x$ (xi) $2x - \mathbf{y} = 0$ (xii) $2x - \mathbf{y} = 2$

(xiii) $x - 3\mathbf{y} + 1 = 0$ (xiv) $3x - 2\mathbf{y} + 1 = 0$

3. Are the following lines (i) parallel to x-axis (ii) parallel to y-axis?

(i) $2x - 1 = 3$ (ii) $x + 2 = -1$ (iii) $2\mathbf{y} + 3 = 2$

(iv) $x + \mathbf{y} = 0$ (v) $2x - 2\mathbf{y} = 0$

4. Find the value of m and c of the following lines by expressing them in the form y = mx + c.

(a) $2x + 3\mathbf{y} - 1 = 0$ (b) $x - 2\mathbf{y} = -2$ (c) $3x + \mathbf{y} - 1 = 0$

(d) $2x - \mathbf{y} = 7$ (e) $3 - 2x + \mathbf{y} = 0$ (f) $2x = \mathbf{y} + 3$

5. Verify whether the following point lies on the line $2x - \mathbf{y} + 1 = 0$ or not.

(i) (2, 3) (ii) (0, 0) (iii) (–1, 1)

(iv) (2, 5) (v) (5, 3)

8.2 Conversion Graphs

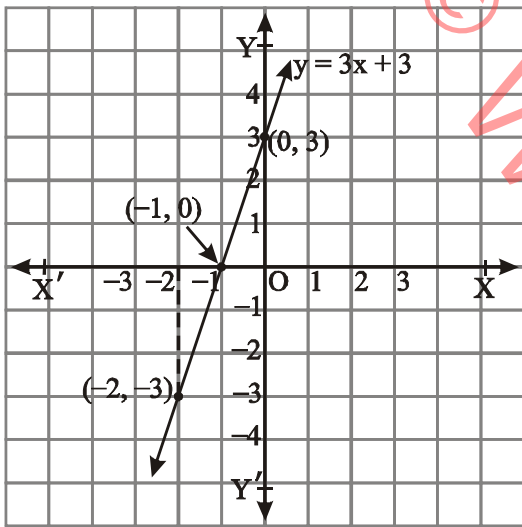
8.2.1 To Interpret Conversion Graph

In this section we shall consider conversion graph as a linear graph relating to two quantities which are in direct proportion.

Let $\mathbf{y} = \mathbf{f}(x)$ be an equation in two variables x and y.

We demonstrate the ordered pairs which lie on the graph of the equation $\mathbf{y} = 3x + 3$ are tabulated below:

x	... 0	–1	–2 ...
y	... 3	0	–3 ...
(x, y)	... (0, 3)	(–1, 0)	(–2, –3) ...

By plotting the points in the plane corresponding to the ordered pairs (0, 3), (–1, 0) and (–2, –3) etc, we form the graph of the equation $\mathbf{y} = 3x + 3$.

8.2.2 Reading a Given Graph

From the graph of **y** = 3x + 3 as shown above.

(i) for a given value of x we can read the corresponding value of **y** with the help of equation **y** = 3x + 3, and

(ii) for a given value of y we can read the corresponding value of x, by converting equation **y** = 3x + 3 to equation $x = \frac{1}{3}y - 1$ and draw the corresponding conversion graph.

In the conversion graph we express x in terms of y as explained below.

y = 3x + 3

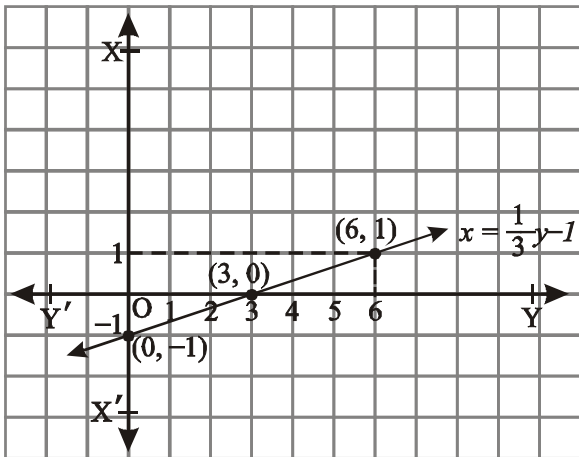
⇒ **y** – 3 = 3x + 3 – 3

⇒ **y** – 3 = 3x or 3x = **y** – 3

⇒ $x = \frac{1}{3}y - 1$, where x is expressed in terms of **y**.

We tabulate the values of the dependent variable x at the values of **y**.

y	... 3	0	6 ...
x	... 0	-1	1 ...
(y, x)	... (3, 0)	(0, -1)	(6, 1) ...

The conversion graph of x with respect to y is displayed as below:

8.2.3 Reading the Graphs of Conversion

(a) Example: (Kilometre (Km) and Mile (M) Graphs)

To draw the graph between kilometre (Km) and Miles (M), we use the following relation:

One kilometre = 0.62 miles, (approximately)

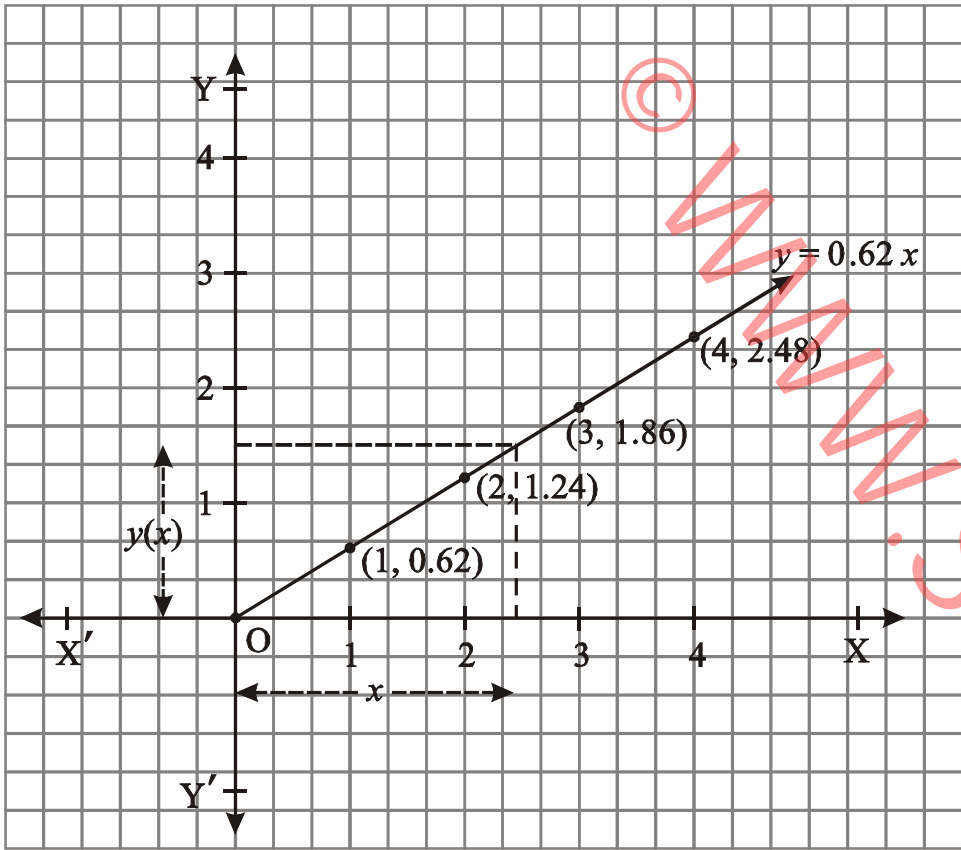
and one mile = 1.6 km (approximately)

(i) The relation of mile against kilometre is given by the linear equation,

$$y = 0.62\,x,$$

If y is a mile and x, a kilometre, then we tabulate the ordered pairs (x, **y**) as below;

x	0	1	2	3	4 ...
y	0	0.62	1.24	1.86	2.48 ...

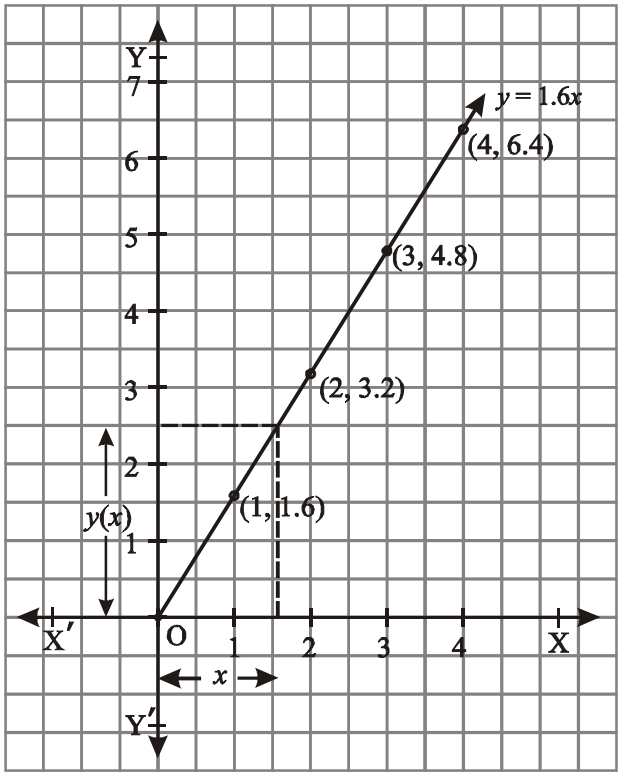
The ordered pairs **(x, y)** corresponding to **y = 0.62x** are represented in the Cartesian plane. By joining them we get the desired following graph of miles against kilometers.

For each quantity of kilometre x along x-axis there corresponds mile along y-axis.

(ii) The **conversion graph** of kilometre against mile is given by

$y = 1.6x$ (approximately)

If y represents kilometres and x a mile, then the values x and y are tabulated as:

x	0	1	2	3	4 ...
y	0	1.6	3.2	4.8	6.4 ...

We plot the points in the xy-plane corresponding to the ordered pairs. (0, 0), (1, 1.6), (2, 3.2), (3, 4.8) and (4, 6.4) as shown in figure.

By joining the points we actually find the conversion graph of kilometres against miles.

(b) Conversion Graph of Hectares and Acres

(i) The relation between Hectare and Acre is defined as:

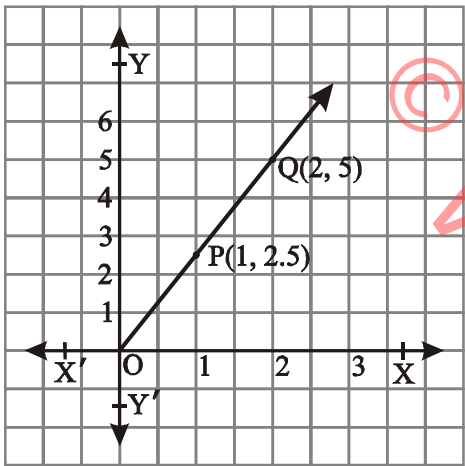
Hectare = $\frac{640}{259}$ Acres

= 2.5 Acres (approximately)

In case when hectare = x and acre = y, then relation between them is given by the equation, $y = 2.5x$

If x is represented as hectare along the horizontal axis and y as Acre along y-axis, the values are tabulated below:

x	0	1	2	3	4 ...
y	0	2.5	5.0	7.5	10 ...

The ordered pairs (0,0), (1, 2.5), (2,5) etc., are plotted as points in the xy-plane as below and by joining the points the required graph is obtained:

(ii) Now the **conversion graph** is Acre = $\frac{1}{2.5}$ Hectare is simplified as,

Acre = $\frac{10}{25}$ Hectare

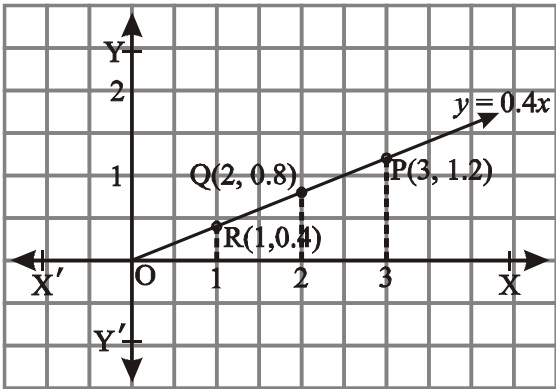
= 0.4 Hectare (approximately)

If Acre is measured along x-axis and hectare along y-axis then

y = 0.4x

The ordered pairs are tabulated in the following table,

x	0	1	2	3 ...
y	0	0.4	0.8	1.2 ...

The corresponding ordered pairs (0, 0), (1, 0.4), (2, 0.8) etc., are plotted in the xy-plane, join of which will form the graph of (b)-ii as a conversion graph of (a)-i:

(c) Conversion Graph of Degrees Celsius and Degrees Fahrenheit

(i) The relation between degree Celsius (C) and degree Fahrenheit (F) is given by

F = $\frac{9}{5}$C + 32

The values of F at C = 0 is obtained as

F = $\frac{9}{5}$ x 0 + 32 = 0 + 32 = 32

Similarly,

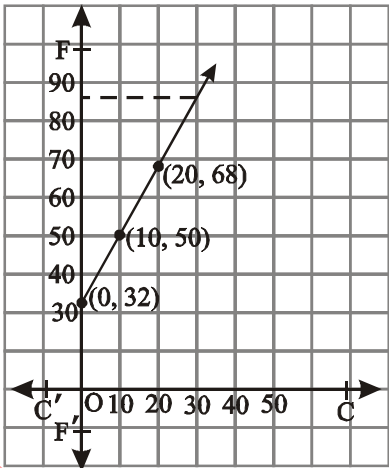
F = $\frac{9}{5}$ x 10 + 32 = 18 + 32 = 50,

F = $\frac{9}{5}$ x 20 + 32 = 36 + 32 = 68,

F = $\frac{9}{5}$ x 100 + 32 = 180 + 32 = 212

We tabulate the values of C and F.

C	0°	10°	20°	50°	100° ...
F	32°	50°	68°	122°	212° ...

The conversion graph of F with respect to C is shown in figure.

10^0 = length of square

Note from the graph that the value of C corresponding to

(i) F = 86° is C = 30° and (ii) F = 104° is C = 40°.

(ii) Now we express C in terms of F for the conversion graph of C with respect to F as below:

C = $\frac{5}{9}$(F – 32)

The values for F = 68° and F = 176° are

$$C = \frac{5}{9}(68 - 32) = \times 36 = 20^0$$

and

$$C = \frac{5}{9}(176 - 32) = \frac{5}{9}(144) = 5 \times 16 = 80^0$$

Find out at what temperature will the two readings be same?

i.e., $F = \frac{9}{5}C + 32$

$$\Rightarrow \left(\frac{9}{5} - 1\right) C = -32 \Rightarrow \frac{4}{5} C = -32 \Rightarrow C = \frac{-32 \times 5}{4} = -40$$

To verify at C = –40, we have

$$F = \frac{9}{5} \times (-40) + 32 = 9(-8) + 32 = -72 + 32 = -40^0$$

(d) Conversion Graph of US and Pakistani Currency

The Daily News, on a particular day informed the conversion rate of Pakistani currency to the US$ currency as,

1 US$ = 66.46 Rupees

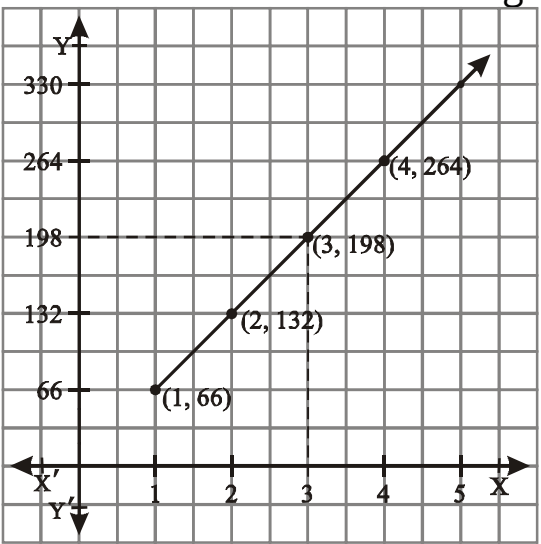
If the Pakistani currency **y** is an expression of US$ $\boldsymbol{x}$, expressed under the rule

$\mathbf{y} = 66.46\boldsymbol{x} = 66\boldsymbol{x}$ (approximately)

then draw the conversion graph.

We tabulate the values as below.

$\boldsymbol{x}$	1	2	3	4 ...
y	66	132	198	264 ...

Plotting the points corresponding to the ordered pairs ($\boldsymbol{x}$, **y**) from the above table and joining them provides the currency linear graph of rupees against dollars as shown in the figure.

Conversion graph $x = \frac{1}{66}$ y of y = 66x can be shown by interchanging x-axis to y-axis and vice versa.

EXERCISE 8.2

1. Draw the conversion graph between litres and gallons using the relation 9 litres = 2 gallons (approximately), and taking litres along horizontal axis and gallons along vertical axis. From the graph, read
   (i) the number of gallons in 18 litres
   (ii) the number of litres in 8 gallons.
2. On 15.03.2008 the exchange rate of Pakistani currency and Saudi Riyal was as under:

   1 S. Riyal = 16.70 Rupees

   If Pakistani currency y is an expression of S. Riyal x, expressed under the rule $y = 16.70x$, then draw the conversion graph between these two currencies by taking S. Riyal along x-axis.
3. Sketch the graph of each of the following lines.

   (a) $\boldsymbol{x} - 3\mathbf{y} + 2 = 0$ (b) $3\boldsymbol{x} - 2\mathbf{y} - 1 = 0$ (c) $2\mathbf{y} - \boldsymbol{x} + 2 = 0$
   (d) $\mathbf{y} - 2\boldsymbol{x} = 0$ (e) $3\mathbf{y} - 1 = 0$ (f) $\mathbf{y} + 3\boldsymbol{x} = 0$
   (g) $2\boldsymbol{x} + 6 = 0$
4. Draw the graph for following relations.
   (i) One mile = 1.6 km (ii) One Acre = 0.4 Hectare
   (iii) $F = \frac{9}{5}C + 32$ (iv) One Rupee = $\frac{1}{86}$ \$

8.3 Graphical Solution of Linear Equations in two Variables

We solve here simultaneous linear equations in two variables by graphical method.

Let the system of equations be,

$2\boldsymbol{x} - \mathbf{y} = 3,$ ...... (i)

$\boldsymbol{x} + 3\mathbf{y} = 3.$ ...... (ii)

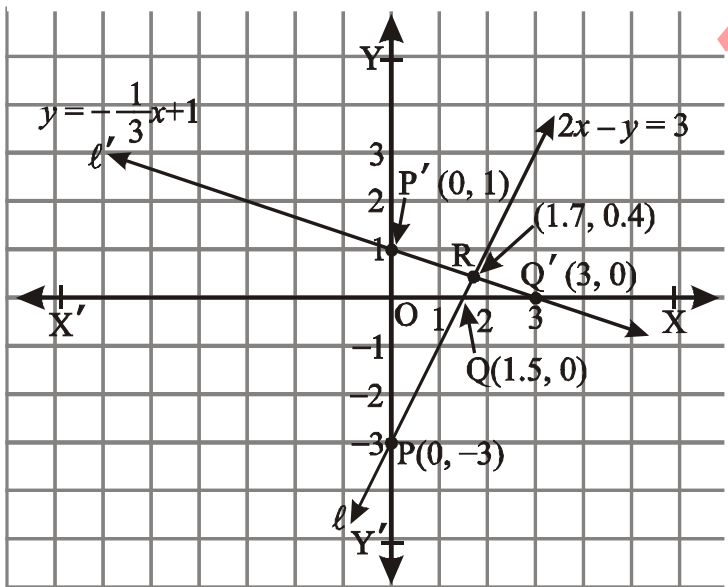
Table of Values

$y = 2x - 3$

x	... 0	1.5 ...
y	... –3	0 ...

$y = -\frac{1}{3}x + 1$

x	... 0	3	...
y	... 1	0	...

By plotting the points, we get the following graph.

The solution of the system is the point R where the lines ℓ and ℓ meet at, i.e., R(1·7, 0·4) such that $x = 1.7$ and $y = 0{\cdot}4$.

Example

Solve graphically, the following linear system of two equations in two variables x and y;

$x + 2y = 3$, ...(i)

$x - y = 2$. ......(ii)

Solution

The equations (i) and (ii) are represented graphically with the help of their points of intersection with the coordinate axes of the same co-ordinate plane.

The points of intersections of the lines representing equation (i) and (ii) are given in the following table:

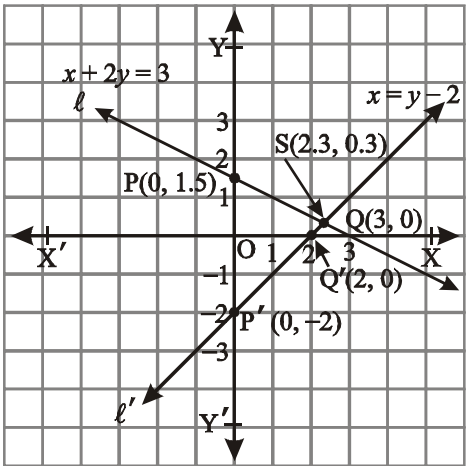
$y = -\frac{x}{2} + \frac{3}{2}$

x	... 0	3 ...
y	... 1.5	0 ...

$y = x - 2$

x	... 0	2 ...
y	... –2	0 ...

The points P(0, 1·5) and Q(3, 0) of equation (i) are plotted in the plane and the corresponding line ℓ: $x + 2y = 3$ is traced by joining P and Q.

Similarly, the line ℓ′: $x - y = 2$ of (ii) is obtained by plotting the points P′(0, –2) and Q′(2, 0) in the plane and joining them to trace the line ℓ′ as below:

The common point S(2.3, 0.3) on both the lines ℓ and ℓ′ is the required solution of the system.

EXERCISE 8.3

Solve the following pair of equations in x and y graphically.

1. $x + y = 0$ and $2x - y + 3 = 0$
2. $x - y + 1 = 0$ and $x - 2y = -1$
3. $2x + y = 0$ and $x + 2y = 2$
4. $x + y - 1 = 0$ and $x - y + 1 = 0$
5. $2x + y - 1 = 0$ and $x = -y$

REVIEW EXERCISE 8

1. Choose the correct answer.

2. Identify the following statements as True or False.
   - (i) The point O(0, 0) is in quadrant II. ......
   - (ii) The point P(2, 0) lies on x-axis. ......
   - (iii) The graph of $x = -2$ is a vertical line. ......
   - (iv) $3 - y = 0$ is a horizontal line. ......
   - (v) The point Q(–1, 2) is in quadrant III. ......
   - (vi) The point R(–1, –2) is in quadrant IV. ......
   - (vii) $y = x$ is a line on which origin lies. ......
   - (viii) The point P(1, 1) lies on the line $x + y = 0$. ......
   - (ix) The point S(1, –3) lies in quadrant III. ......
   - (x) The point R(0, 1) lies on the x-axis. ...
3. Draw the following points on the graph paper.
   (–3, –3), (–6, 4), (4, –5), (5, 3)
4. Draw the graph of the following
   - (i) $x = -6$ (ii) $y = 7$
   - (iii) $x = \frac{5}{2}$ (iv) $y = -\frac{9}{2}$
   - (v) $y = 4x$ (vi) $y = -2x + 1$
5. Draw the following graph.
   - (i) $y = 0.62x$ (ii) $y = 2.5x$
6. Solve the following equations graphically.
   - (i) $x - y = 1$, $x + y = \frac{1}{2}$
   - (ii) $x = 3y$, $2x - 3y = -6$
   - (iii) $(x + y) = 2$, $\frac{1}{2}(x - y) = -1$

SUMMARY

- An ordered pair is a pair of elements in which elements are written in specific order.
- The plane framed by two straight lines perpendicular to each other is called cartesian plane and the lines are called coordinate axes.
- The point of intersection of two coordinate axes is called origin.
- There is a one-to-one correspondence between ordered pair and a point in Cartesian plane and vice versa.
- Cartesian plane is also known as coordinate plane.
- Cartesian plane is divided into four quadrants.
- The x-coordinate of a point is called abcissa and y-coordinate is called ordinate.
- The set of points which lie on the same line are called collinear points.

CHAPTER 9

INTRODUCTION TO COORDINATE GEOMETRY

Animation 9.1: Algebraic Manipulation
Source & Credit: eLearn.punjab

Students Learning Outcomes

After studying this unit, the students will be able to:

- Define coordinate geometry.
- Derive distance formula to calculate distance between two points given in Cartesian plane.
- Use distance formula to find distance between two given points.
- Define collinear points. Distinguish between collinear and non-collinear points.
- Use distance formula to show that given three (or more) points are collinear.
- Use distance formula to show that the given three non-collinear points form
  - an equilateral triangle,
  - an isosceles triangle,
  - a right angled triangle,
  - a scalene triangle.
- Use distance formula to show that given four non-collinear points form
  - a square,
  - a rectangle,
  - a parallelogram.
- Recognize the formula to find the midpoint of the line joining two given points.
- Apply distance and mid point formulae to solve/verify different standard results related to geometry.

9.1 Distance Formula

9.1.1 Coordinate Geometry

The study of geometrical shapes in a plane is called plane geometry. Coordinate geometry is the study of geometrical shapes in the Cartesian plane (coordinate plane). We know that a plane is divided into four quadrants by two perpendicular lines called the axes intersecting at origin. We have also seen that there is one to one correspondence between the points of the plane and the ordered pairs in R × R.

9.1.2 Finding Distance between two points

Let P(x_1, y_1) and Q(x_2, y_2) be two points in the coordinate plane where d is the length of the line segment PQ. i.e. ,$|PQ| = d$.

The line segments MQ and LP parallel to y-axis meet x-axis at points M and L, respectively with coordinates M(x_2, 0) and L(x_1, 0).

The line-segment PN is parallel to x-axis.

In the right triangle PNQ,

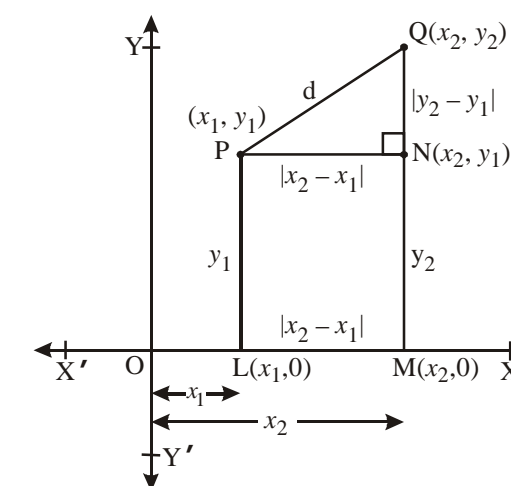
$|NQ| = |y_2 - y_1|$ and $|PN| = |x_2 - x_1|$.

Using Pythagoras Theorem

$$(\overline{PQ})^2 = (\overline{PN})^2 + (\overline{QN})^2$$

$$\Rightarrow d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$\Rightarrow d^2 = \pm\sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$

$$\boxed{\text{Thus } d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}}, \text{ since } d > 0 \text{ always.}$$

9.1.3 Use of Distance Formula

The use of distance formula is explained in the following examples.

Example 1

Using the distance formula, find the distance between the points.

(i) P(1, 2) and Q(0, 3) (ii) S(–1, 3) and R(3, –2)
(iii) U(0, 2) and V(–3, 0) (iv) P′(1, 1) and Q′(2, 2)

Solution

(i) $|PQ| = \sqrt{(0-1)^2 + (3-2)^2}$

$= \sqrt{(-1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$

(ii) $|SR| = \sqrt{(3-(-1))^2 + (-2-3)^2}$

$= \sqrt{(3+1)^2 + (-5)^2} = \sqrt{16+25} = \sqrt{41}$

(iii) $|UV| = \sqrt{(-3-0)^2 + (0-2)^2}$

$= \sqrt{(-3)^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$

(iv) $|P'Q'| = \sqrt{(2-1)^2 + (2-1)^2}$

$= \sqrt{1+1} = \sqrt{2}$

EXERCISE 9.1

1. Find the distance between the following pairs of points.

(a) A(9, 2), B(7, 2) (b) A(2, –6), B(3, –6)
(c) A(–8, 1), B(6, 1) (d) A(–4, $\sqrt{2}$), B(–4, –3),
(e) A(3, –11), B(3, –4) (f) A(0, 0), B(0, –5)

2. Let P be the point on x-axis with x-coordiante a and Q be the point on y-axis with y-coordinate b as given below. Find the distance between P and Q.

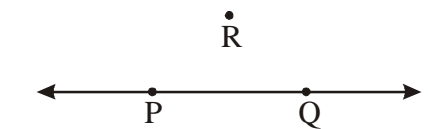
(i) $a = 9, b = 7$ (ii) $a = 2, b = 3$ (iii) $a = -8, b = 6$
(iv) $a = -2, b = -3$ (v) $a = \sqrt{2}, b = 1$ (vi) $a = -9, b = -4$

9.2 Collinear Points

9.2.1 Collinear or Non-collinear Points in the Plane

Two or more than two points which lie on the same straight line are called collinear points with respect to that line; otherwise they are called non-collinear.

Let m be a line, then all the points on line m are collinear.

In the given figure, the points P and Q are collinear with respect to the line m and the points P and R are not collinear with respect to it.

9.2.2 Use of Distance Formula to show the Collinearity of Three or more Points in the Plane

Let P, Q and R be three points in the plane. They are called collinear if $|PQ| + |QR| = |PR|$, otherwise will be non colliner.

Example

Using distance formula show that the points
(i) P(–2, –1), Q(0, 3) and R(1, 5) are collinear.
(ii) The above points P, Q, R and S(1, –1) are not collinear.

Solution

(i) By using the distance formula, we find

$$|PQ| = \sqrt{(0+2)^2 + (3+1)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

$$|QR| = \sqrt{(1-0)^2 + (5-3)^2} = \sqrt{1+4} = \sqrt{5}$$

and $$|PR| = \sqrt{(1+2)^2 + (5+1)^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$$

Since $|PQ| + |QR| = 2\sqrt{5} + \sqrt{5} = 3\sqrt{5} = |PR|$,

therefore, the points P, Q and R are collinear

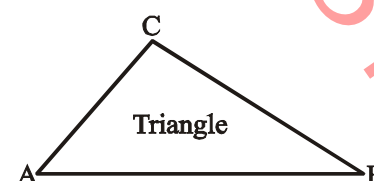
(ii) $|PS| = \sqrt{(-2-1)^2 + (-1+1)^2} = \sqrt{(-3)^2 + 0} = 3$

Since $|QS| = \sqrt{(1-0)^2 + (-1-3)^2} = \sqrt{1+16} = \sqrt{17}$,

and $|PQ| + |QS| \neq |PS|$,

therefore the points P, Q and S are not collinear and hence, the points P, Q, R and S are also not collinear.

A closed figure in a plane obtained by joining three non-collinear points is called a triangle.

In the triangle ABC the non-collinear points A, B and C are the three vertices of the triangle ABC. The line segments AB, BC and CA are called sides of the triangle.

9.2.3 Use of Distance Formula to Different Shapes of a Triangle

We expand the idea of a triangle to its different kinds depending on the length of the three sides of the triangle as:

(i) Equilateral triangle (iii) Isosceles triangle
(ii) Right angled triangle (iv) Scalene triangle

We discuss the triangles (i) to (iv) in order.

(i) Equilateral Triangle

If the lengths of all the three sides of a triangle are same, then the triangle is called an equilateral triangle.

Example

The triangle OPQ is an equilateral triangle since the points O(0, 0), $P\left(\frac{1}{\sqrt{2}}, 0\right)$ and $Q\left(\frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\right)$ are not collinear, where

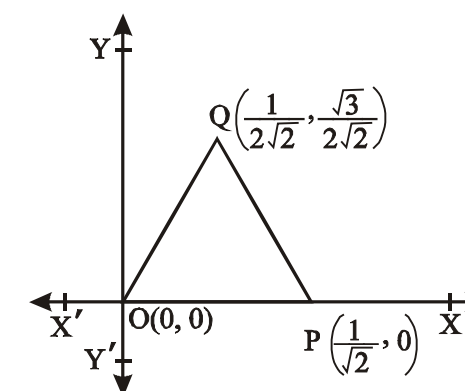
$$|OP| = \frac{1}{\sqrt{2}}$$

$$|QO| = \sqrt{\left(0-\frac{1}{2\sqrt{2}}\right)^2 + \left(0-\frac{\sqrt{3}}{2\sqrt{2}}\right)^2} = \sqrt{\frac{1}{8}+\frac{3}{8}} = \sqrt{\frac{4}{8}} = \sqrt{\frac{1}{2}}$$

and

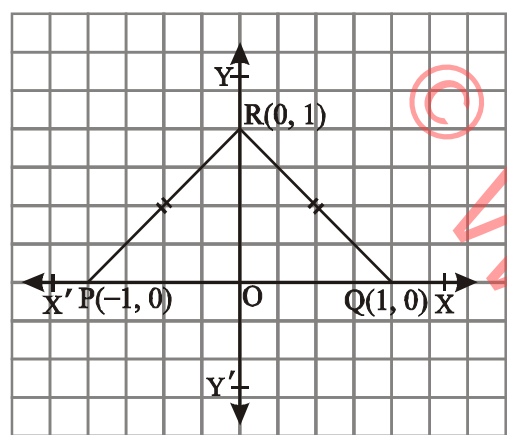
$$|PQ| = \sqrt{\left(\frac{1}{2\sqrt{2}}-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}}{2\sqrt{2}}-0\right)^2} = \sqrt{\left(\frac{1-2}{2\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{1}{8}+\frac{3}{8}} = \sqrt{\frac{4}{8}} = \sqrt{\frac{1}{2}}$$

i.e., $|OP| = |QO| = |PQ| = \frac{1}{\sqrt{2}}$, a real number and the points O(0, 0), $Q\left(\frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\right)$ and $P\left(\frac{1}{\sqrt{2}}, 0\right)$ are not collinear. Hence the triangle OPQ is equilateral.

(ii) An Isosceles Triangle

An isosceles triangle PQR is a triangle which has two of its sides with equal length while the third side has a different length.

Example

The triangle PQR is an isosceles triangle as for the non-collinear points P(–1, 0), Q(1, 0) and R (0, 1) shown in the following figure,

$$|PQ| = \sqrt{(1-(-1))^2 + (0-0)^2} = \sqrt{(1+1)^2 + 0} = \sqrt{4} = 2$$
$$|QR| = \sqrt{(0-1)^2 + (1-0)^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$$
$$|PR| = \sqrt{(0-(-1))^2 + (1-0)^2} = \sqrt{1+1} = \sqrt{2}$$

Since $|QR| = |PR| = \sqrt{2}$ and $|PQ| = 2 \neq \sqrt{2}$ so the non-collinear points P, Q, R form an isosceles triangle PQR.

(iii) Right Angled Triangle

A triangle in which one of the angles has measure equal to 90^0 is called a right angle triangle.

Example

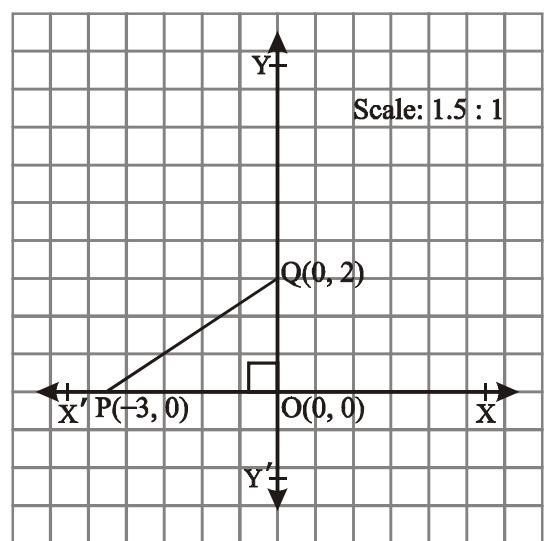
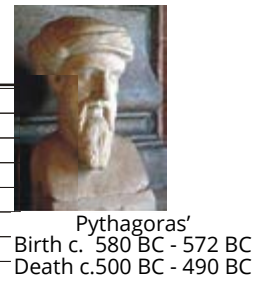
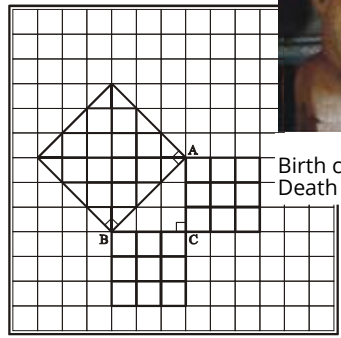
Let O(0, 0), P(–3, 0) and Q(0, 2) be three non-collinear points. Verify that triangle OPQ is right-angled.

$$|OQ| = \sqrt{(0-0)^2 + (2-0)^2} = \sqrt{2^2} = 2$$
$$|OP| = \sqrt{(-3)^2 + 0^2} = \sqrt{9} = 3$$
$$|PQ| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$$

Visual proof of pythagoras' therom

In right angle triangle ABC

|AB|² = |BC|² + |CA|²

Pythagoras'
Birth c. 580 BC - 572 BC
Death c.500 BC - 490 BC

Here 1.5 square block = 1 unit length

Now $|OQ|^2 + |OP|^2 = (2)^2 + (3)^2 = 13$ and $|PQ|^2 = 13$

Since $|OQ|^2 + |OP|^2 = |PQ|^2$, therefore $\angle POQ = 90^\circ$

Hence the given non-collinear points form a right triangle.

(iv) Scalene Triangle

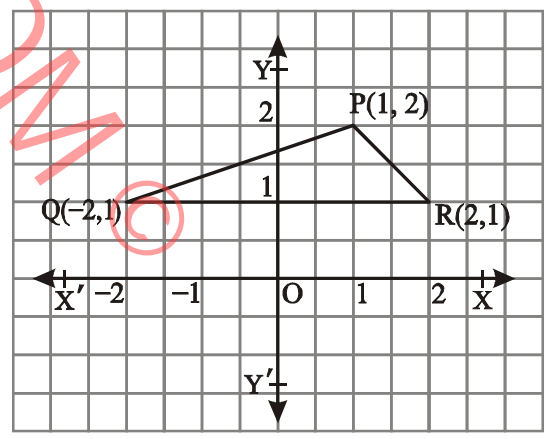
A triangle is called a scalene triangle if measures of all the three sides are different.

Example

Show that the points P(1, 2), Q(–2, 1) and R(2, 1) in the plane form a scalene triangle.

Solution

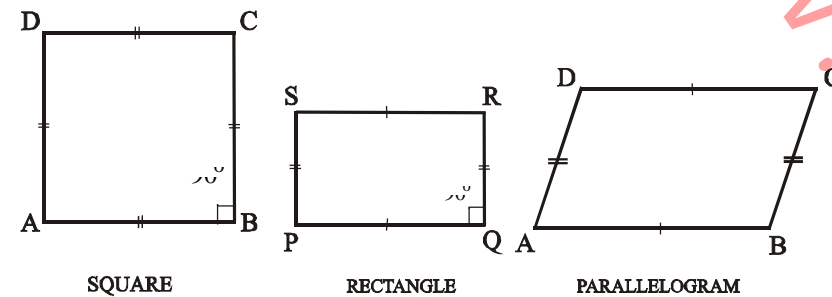
$$|PQ| = \sqrt{(-2-1)^2 + (1-2)^2} = \sqrt{(-3)^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$$
$$|QR| = \sqrt{(2+2)^2 + (1-1)^2} = \sqrt{4^2 + 0^2} = \sqrt{4^2} = 4$$

and $|PR| = \sqrt{(2-1)^2 + (1-2)^2} = \sqrt{1^2 + (-1)^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

Hence $|PQ| = \sqrt{10}$, $|QR| = 4$ and $|PR| = \sqrt{2}$

The points P, Q and R are non-collinear since, $|PQ| + |QR| > |PR|$

Thus the given points form a scalene triangle.

9.2.4 Use of distance formula to show that four non-collinear points form a square, a rectangle and a parallelogram

We recognize these three figures as below

(a) Using Distance Formula to show that given four Non-Collinear Points form a Square

A square is a closed figure in the plane formed by four non-collinear points such that lengths of all sides are equal and measure of each angle is 90^0.

Example

If A(2, 2), B(2, –2), C(–2, –2) and D (–2, 2) be four non-collinear points in the plane, then verify that they form a square ABCD.

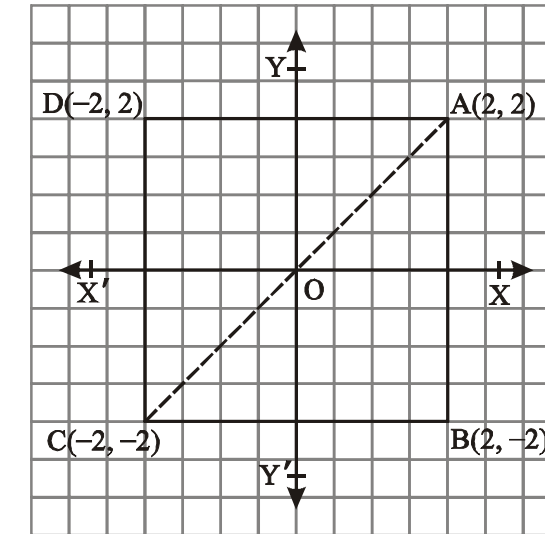
Solution

$$|AB| = \sqrt{(2-2)^2 + (-2-2)^2} = \sqrt{0^2 + (-4)^2} = \sqrt{16} = 4$$

$$|BC| = \sqrt{(-2-2)^2 + (-2+2)^2} = \sqrt{(-4)^2 + 0^2} = \sqrt{16} = 4$$

$$|CD| = \sqrt{(-2-(-2))^2 + (2-(-2))^2}$$

$$= \sqrt{(-2+2)^2 + (2+2)^2} = \sqrt{0+16} = \sqrt{16} = 4$$

$$|DA| = \sqrt{(2+2)^2 + (2-2)^2} = \sqrt{(+4)^2 + 0} = \sqrt{16} = 4,$$

Hence AB = BC = CD = DA = 4.

Also $|AC| = \sqrt{(-2-2)^2 + (-2-2)^2} = \sqrt{(-4)^2 + (-4)^2} = \sqrt{16+16} = \sqrt{32}$

$= 4\sqrt{2}$

Now $|AB|^2 + |BC|^2 = |AC|^2$, therefore $\angle ABC = 90^0$

Hence the given four non collinear points form a square.

(b) Using Distance Formula to show that given four Non-Collinear Points form a Rectangle

A figure formed in the plane by four non-collinear points is called a rectangle if,

(i) its opposite sides are equal in length;

(ii) the angle at each vertex is of measure 90^o.

Example

Show that the points A(–2, 0), B(–2, 3), C(2, 3) and D(2, 0) form a rectangle.

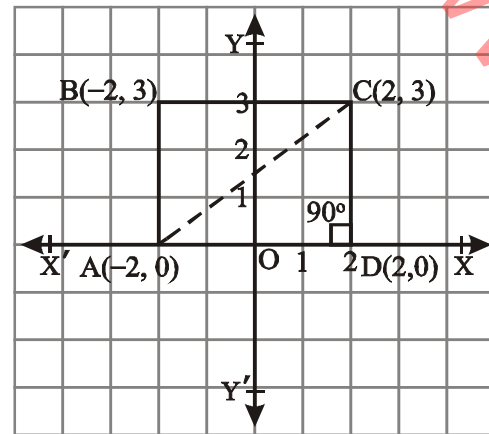
Solution

Using distance formula,

$$|AB| = \sqrt{(-2+2)^2 + (3-0)^2} = \sqrt{0+9} = \sqrt{9} = 3$$

$$|DC| = \sqrt{(2-2)^2 + (3-0)^2} = \sqrt{0+9} = \sqrt{9} = 3$$

$|AD| = \sqrt{(2+2)^2+(0-0)^2} = \sqrt{16+0} = 4$

$|BC| = \sqrt{(2+2)^2+(3-3)^2} = \sqrt{16+0} = \sqrt{16} = 4$

Since $|AB| = |DC| = 3$ and $|AD| = |BC| = 4$, therefore, opposite sides are equal.

Also $|AC| = \sqrt{(2+2)^2+(3-0)^2} = \sqrt{16+9} = \sqrt{25} = 5$

Now $|AD|^2 + |DC|^2 = (4)^2 + (3)^2 = 25$, and $= |AC|^2 = (5)^2 = 25$

Since $|AD|^2 + |DC|^2 = |AC|^2$,

therefore $m\angle ADC = 90^\circ$

Hence the given points form a rectangle.

(c) Use of Distance Formula to show that given four Non-Collinear Points Form a Parallelogram

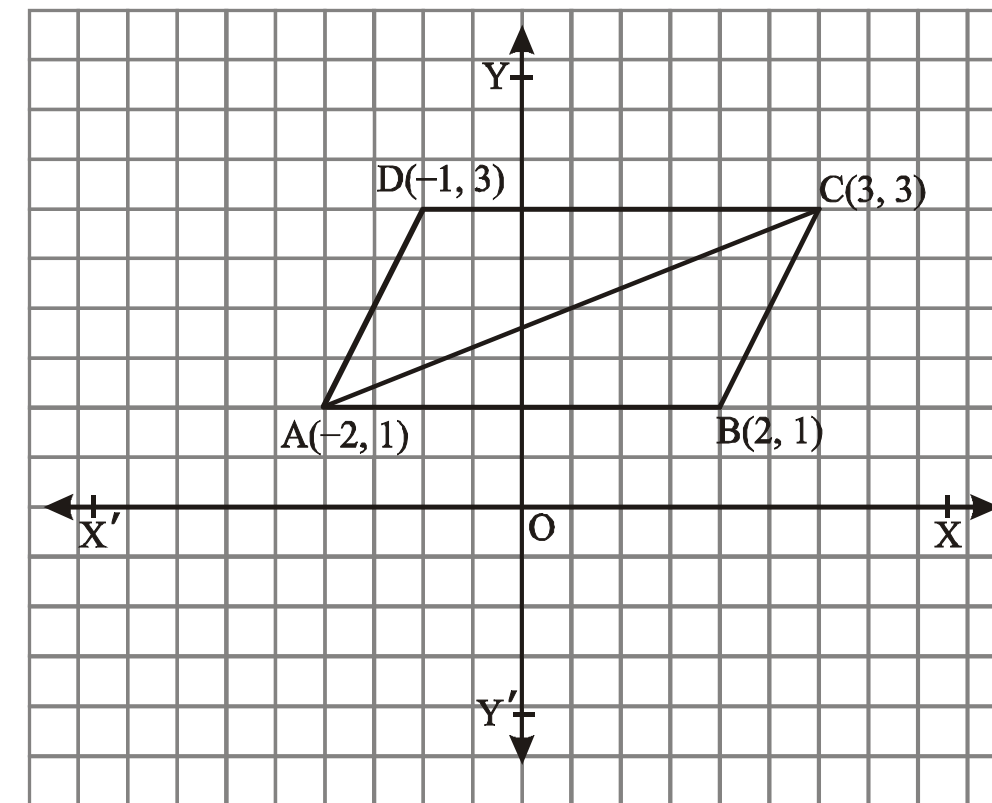
Definition

A figure formed by four non-collinear points in the plane is called a **parallelogram** if

(i) its opposite sides are of equal length

(ii) its opposite sides are parallel

Example

Show that the points A(–2, 1), B(2, 1), C(3, 3) and D(–1, 3) form a parallelogram.

By distance formula,

$|AB| = \sqrt{(2+2)^2+(1-1)^2} = \sqrt{4^2+0} = \sqrt{16} = 4$

$|CD| = \sqrt{(3+1)^2+(3-3)^2} = \sqrt{4^2+0} = \sqrt{16} = 4$

$|AD| = \sqrt{(-1+2)^2+(3-1)^2} = \sqrt{1^2+2^2} = \sqrt{1+4} = \sqrt{5}$

$|BC| = \sqrt{(3-2)^2+(3-1)^2} = \sqrt{1^2+2^2} = \sqrt{5}$

Since $|AB| = |CD| = 4$ and $|AD| = |BC| = \sqrt{5}$

Hence the given points form a parallelogram.

EXERCISE 9.2

1. Show whether the points with vertices (5, –2), (5, 4) and (–4, 1) are vertices of an equilateral triangle or an isosceles triangle?
2. Show whether or not the points with vertices (–1, 1), (5, 4), (2, –2) and (–4, 1) form a square?
3. Show whether or not the points with coordinates (1, 3), (4, 2) and

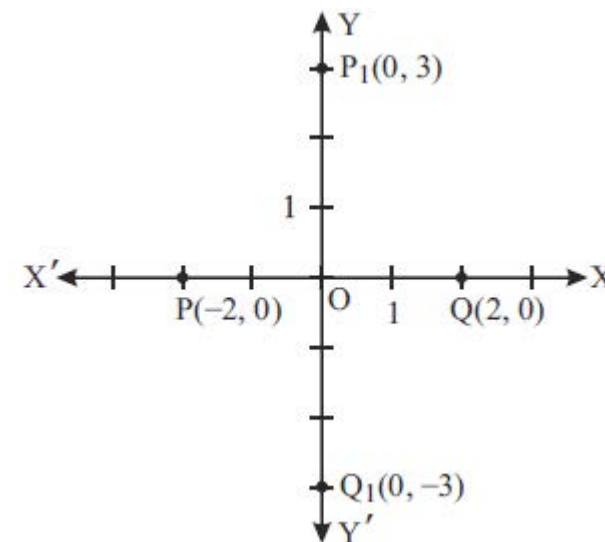
(–2, 6) are vertices of a right triangle?

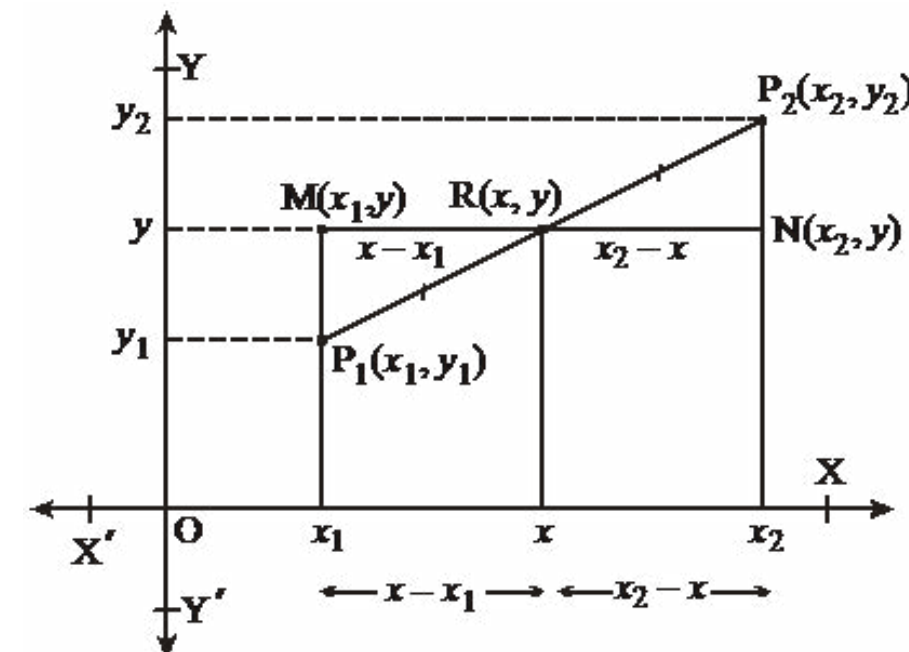
4. Use the distance formula to prove whether or not the points (1, 1), (–2, –8) and (4, 10) lie on a straight line?
5. Find k, given that the point (2, k) is equidistant from (3, 7) and (9, 1).
6. Use distance formula to verify that the points A(0, 7), B(3, –5), C(–2, 15) are collinear.
7. Verify whether or not the points O(0, 0), A($\sqrt{3}$, 1), B($\sqrt{3}$, –1) are vertices of an equilateral triangle.
8. Show that the points A(–6, –5), B(5, –5), C(5, –8) and D(–6, –8) are vertices of a rectangle. Find the lengths of its diagonals. Are they equal?
9. Show that the points M(–1, 4), N(–5, 3), P(1 –3) and Q(5, –2) are the vertices of a parallelogram.
10. Find the length of the diameter of the circle having centre at C(–3, 6) and passing through P(1, 3).

9.3 Mid-Point Formula

9.3.1 Recognition of the Mid-Point

Let P(–2, 0) and Q(2, 0) be two points on the x-axis. Then the origin O(0, 0) is the mid point of P and Q, since |OP| = 2 = |OQ| and the points P, O and Q are collinear.

Similarly the origin is the mid-point of the points P_1(0, 3) and Q_1(0, –3) since $|OP_1| = 3 = |OQ_1|$ and the points P_1, O and Q_1 are collinear.

Recognition of the Mid-Point Formula for any two Points in the Plane

Let P1(x_1, y_1) and $P_2(x_2, y_2)$ be any two points in the plane and R(x, y) be a mid-point of points P_1 and P_2 on the line-segment P_1P_2 as shown in the figure below.

If line-segment MN, parallel to x-axis, has its mid-point R(x, y), then, $x_2 - x = x - x_1$

$\Rightarrow \quad 2x = x_1 + x_2 \qquad \Rightarrow x = \dfrac{x_1 + x_2}{2}$

Similarly, $\quad y = \dfrac{y_1 + y_2}{2}$

Thus the point R(x, y) = $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$ is the mid-point of the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

9.3.2 Verification of the Mid-PointFormula

$$|P_1R| = \sqrt{\left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - y_1\right)^2}$$

$$= \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \frac{1}{2}|P_1P_2|$$

and
$$|P_2R| = \sqrt{\left(\frac{x_1+x_2}{2}-x_2\right)^2+\left(\frac{y_1+y_2}{2}-y_2\right)^2}$$
$$= \sqrt{\left(\frac{x_1+x_2-2x_2}{2}\right)^2+\left(\frac{y_1+y_2-2y_2}{2}\right)^2}$$
$$= \sqrt{\left(\frac{x_1-x_2}{2}\right)^2+\left(\frac{y_1-y_2}{2}\right)^2}$$
$$= \sqrt{\left(\frac{x_2-x_1}{2}\right)^2+\left(\frac{y_2-y_1}{2}\right)^2}$$
$$= \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$\Rightarrow \quad |P_2R| = |P_1R| = \frac{1}{2}|P_1P_2|$$

Thus it verifies that $R\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ is the mid-point of the line segment P_1RP_2 which lies on the line segment since,

$$|P_1R| + |P_2R| = |P_1P_2|$$

> If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the plane, then the mid-point $R(x, y)$ of the line segment PQ is
> $$R(x, y) = R\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Example 1

Find the mid-point of the line segment joining A(2, 5) and B(–1, 1).

Solution

If $R(x, y)$ is the desired mid-point then,

$$x = \frac{2-1}{2} = \frac{1}{2} \quad \text{and} \quad y = \frac{5+1}{2} = \frac{6}{2} = 3$$

Hence $R(x, y) = R\left(\frac{1}{2}, 3\right)$

Example 2

Let P(2, 3) and $Q(x, y)$ be two points in the plane such that R(1, –1) is the mid-point of the points P and Q. Find x and y.

Solution

Since R(1, –1) is the mid point of P(2, 3) and $Q(x, y)$ then

$$1 = \frac{x+2}{2} \quad \text{and} \quad -1 = \frac{y+3}{2}$$

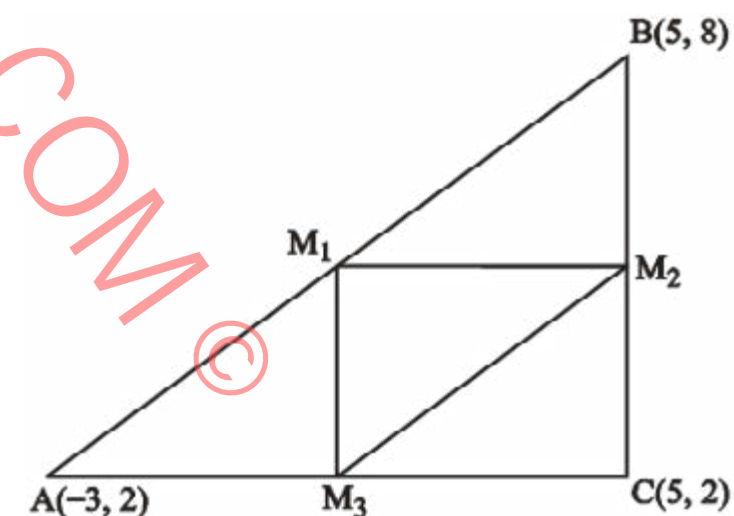
$$\Rightarrow 2 = x+2 \qquad \Rightarrow -2 = y+3$$
$$\Rightarrow x = 0 \qquad \Rightarrow y = -5$$

Example 3

Let ABC be a triangle as shown below. If M_1, M_2 and M_3 are the middle points of the line-segments AB, BC and CA respectively, find the coordinates of M_1, M_2 and M_3. Also determine the type of the triangle $M_1M_2M_3$.

Solution

Mid - point of AB $= M_1\left(\frac{-3+5}{2}, \frac{2+8}{2}\right) = M_1(1, 5)$

Mid - point of BC $= M_2\left(\frac{5+5}{2}, \frac{8+2}{2}\right) = M_2(5, 5)$

and Mid - point of AC = $M_3\left(\dfrac{5-3}{2}, \dfrac{2+2}{2}\right) = M_3(1, 2)$

The triangle $M_1M_2M_3$ has sides with length,

$$|M_1M_2| = \sqrt{(5-1)^2+(5-5)^2} = \sqrt{4^2+0} = 4 \quad \text{....(i)}$$

$$|M_2M_3| = \sqrt{(1-5)^2+(2-5)^2} = \sqrt{(-4)^2+(-3)^2}$$

$$= \sqrt{16+9} = \sqrt{25} = 5 \quad \text{....(ii)}$$

and $|M_1M_3| = \sqrt{(1-1)^2+(2-5)^2} = \sqrt{0^2+(-3)^2} = 3$ ....(iii)

All the lengths of the three sides are different. Hence the triangle $M_1M_2M_3$ is a Scalene triangle

Example 4

Let O(0, 0), A(3, 0) and B(3, 5) be three points in the plane. If M_1 is the mid point of AB and M_2 of OB, then show that $|M_1M_2| = \frac{1}{2}|OA|$.

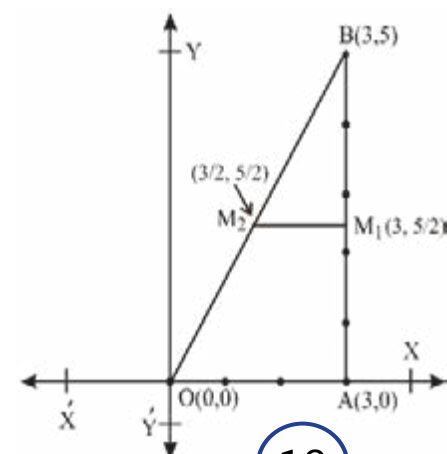
Solution

By the distance formula the distance

$$|OA| = \sqrt{(3-0)^2+(0-0)^2} = \sqrt{3^2} = 3$$

The mid-point of AB is

$$M_1 = M_1\left(\frac{3+3}{2}, \frac{5}{2}\right) = \left(3, \frac{5}{2}\right)$$

Now the mid - point of OB is $M_2 = M_2\left(\dfrac{3+0}{2}, \dfrac{5+0}{2}\right) = \left(\dfrac{3}{2}, \dfrac{5}{2}\right)$

Hence

$$|M_1M_2| = \sqrt{\left(\frac{3}{2}-3\right)^2+\left(\frac{5}{2}-\frac{5}{2}\right)^2} = \sqrt{\left(\frac{-3}{2}\right)^2+0} = \sqrt{\frac{9}{4}+0} = \frac{3}{2} = \frac{1}{2}|OA|$$

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points and their midpoint be $M\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$. Then M

(i) is at equal distance from P and Q
i.e., PM = MQ

(ii) is an interior point of the line segment PQ.

(iii) every point R in the plane at equal distance from P and Q is not their mid-point. For example, the point R(0, 1) is at equal distance from P(–3, 0) and Q(3, 0) but is not their mid-point

i.e. $|RQ| = \sqrt{(0-3)^2+(1-0)^2} = \sqrt{(-3)^2+(1)^2} = \sqrt{9+1} = \sqrt{10}$

$|RP| = \sqrt{(0+3)^2+(1-0)^2} = \sqrt{3^2+1^2} = \sqrt{10}$

and mid-point of P(–3, 0) and Q(3, 0) is (x, y)

Where $x = \dfrac{-3+3}{2} = 0$ and $y = \dfrac{0+0}{2} = 0$.

The point (0, 1) ≠ (0, 0)

(iv) There is a unique midpoint of any two points in the plane.

EXERCISE 9.3

1. Find the mid-point of the line segment joining each of the following pairs of points
   (a) A(9, 2), B(7, 2) (b) A(2, –6), B(3, –6)
   (c) A(–8, 1), B(6, 1) (d) A(–4, 9), B(–4, –3),
   (e) A(3, –11), B(3, –4) (f) A(0, 0), B(0, –5)
2. The end point P of a line segment PQ is (–3, 6) and its mid-point is (5, 8). Find the coordinates of the end point Q.
3. Prove that mid-point of the hypotenuse of a right triangle is equidistant from its three vertices P(–2, 5), Q(1, 3) and R(–1, 0).

4. If O(0, 0), A(3, 0) and B(3, 5) are three points in the plane, find M_1 and M_2 as mid-points of the line segments AB and OB respectively. Find $|M_1M_2|$.
5. Show that the diagonals of the parallelogram having vertices A(1, 2), B(4, 2), C(–1, –3) and D(–4, –3) bisect each other.
[Hint: The mid-points of the diagonals coincide]
6. The vertices of a triangle are P(4, 6), Q(–2, –4) and R(–8, 2). Show that the length of the line segment joining the mid-points of the line segments PR, QR is $\frac{1}{2}$ PQ.

REVIEW EXERCISE 9

1. Choose the correct answer.

2. Answer the following, which is true and which is false.

(i) A line has two end points.
(ii) A line segment has one end point.
(iii) A triangle is formed by three collinear points.
(iv) Each side of a triangle has two collinear vertices.
(v) The end points of each side of a rectangle are collinear.
(vi) All the points that lie on the x-axis are collinear.
(vii) Origin is the only point collinear with the points of both the axes separately.

3. Find the distance between the following pairs of points.

(i) (6, 3), (3, –3) (ii) (7, 5), (1, –1) (iii) (0, 0), (–4, –3)

4. Find the mid-point between following pairs of points.

(i) (6, 6), (4, –2) (ii) (-5, -7), (-7, –5) (iii) (8, 0), (0, –12)

5. Define the following:

(i) Co-ordinate Geometry (ii) Collinear Points
(iii) Non-collinear (iv) Equilateral Triangle
(v) Scalene Triangle (vi) Isosceles Triangle
(vii) Right Triangle (viii) Square

SUMMARY

- If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points and d is the distance between them, then

$$d = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2}$$

- The concept of non-collinearity supports formation of the three-sided and four-sided shapes of the geometrical figures.
- The points P, Q and R are collinear if |PQ| + |QR| = |PR|
- The three points P, Q and R form a triangle if and only if they are non-collinear i.e., |PQ| + |QR| > |PR|
- If |PQ| + |QR| < |PR|, then no unique triangle can be formed by the points P, Q and R.
- Different forms of a triangle i.e., equilateral, isosceles, right angled and scalene are discussed in this unit.
- Similarly, the four-sided figures, square, rectangle and parallelogram are also discussed.

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Introduction to Geometry

Measure angles, prove geometric theorems, and discover how to calculate areas and volumes in this interactive course!

About this Course:

More than 2000 years ago, long before rockets were launched into orbit or explorers sailed around the globe, a Greek mathematician measured the size of the Earth using nothing more than a few facts about lines, angles, and circles. This course will start at the very beginnings of geometry, answering questions like "How big is an angle?" and "What are parallel lines?" and proceed up through advanced theorems and proofs about 2D and 3D shapes.

CHAPTER

10

CONGRUENT TRIANGLES

Animation 10.1: Algebraic Manipulation
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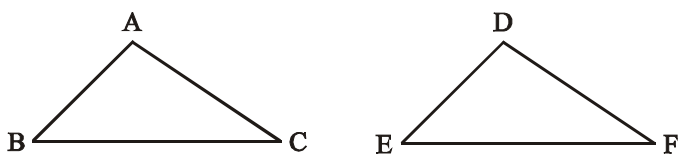
Students Learning Outcomes

After studying this unit, the students will be able to:

- Prove that in any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, then the triangles are congruent.
- Prove that if two angles of a triangle are congruent, then the sides opposite to them are also congruent.
- Prove that in a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, the two triangles are congruent.
- Prove that if in the correspondence of two right-angled triangles, the hypotenuse and one side of one are congruent to the hypotenuses and the corresponding side of the other, then the triangles are congruent.

10.1. Congruent Triangles

Introduction

In this unit before proving the theorems, we will explain what is meant by 1 – 1 correspondence (the symbol used for 1 – 1 correspondence is $\longleftrightarrow$ and congruency of triangles. We shall also state S.A.S. postulate.

Let there be two triangles ABC and DEF. Out of the total six (1 – 1) correspondences that can be established between ΔABC and ΔDEF, one of the choices is explained below.

In the correspondence $\Delta ABC \longleftrightarrow \Delta DEF$ it means

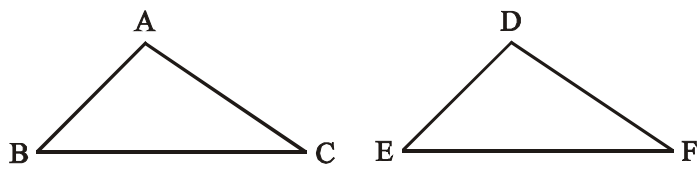
$\angle A \longleftrightarrow \angle D$ ($\angle A$ corresponds to $\angle D$)
$\angle B \longleftrightarrow \angle E$ ($\angle B$ corresponds to $\angle E$)
$\angle C \longleftrightarrow \angle F$ ($\angle C$ corresponds to $\angle F$)

$\overline{AB} \longleftrightarrow \overline{DE}$ ($\overline{AB}$ corresponds to $\overline{DE}$)
$\overline{BC} \longleftrightarrow \overline{EF}$ ($\overline{BC}$ corresponds to $\overline{EF}$)
$\overline{CA} \longleftrightarrow \overline{FD}$ ($\overline{CA}$ corresponds to $\overline{FD}$)

Congruency of Triangles

Two triangles are said to be congruent written symbolically as, $\cong$, if there exists a correspondence between them such that all the corresponding sides and angles are congruent i.e.,

If $\begin{cases} \overline{AB} \cong \overline{DE} \\ \overline{BC} \cong \overline{EF} \\ \overline{CA} \cong \overline{FD} \end{cases}$ and $\begin{cases} \angle A \cong \angle D \\ \angle B \cong \angle E \\ \angle C \cong \angle F \end{cases}$

then $\Delta ABC \cong \Delta DEF$

Note:

(i) These triangles are congruent w.r.t. the above mentioned choice of the (1 – 1) correspondence.

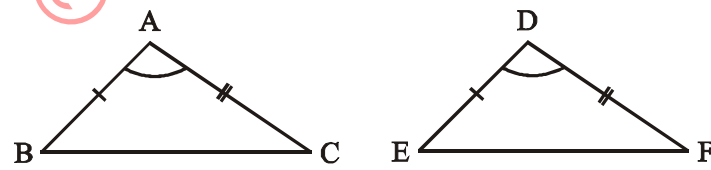
(ii) $\Delta ABC \cong \Delta ABC$

(iii) $\Delta ABC \cong \Delta DEF \Leftrightarrow \Delta DEF \cong \Delta ABC$

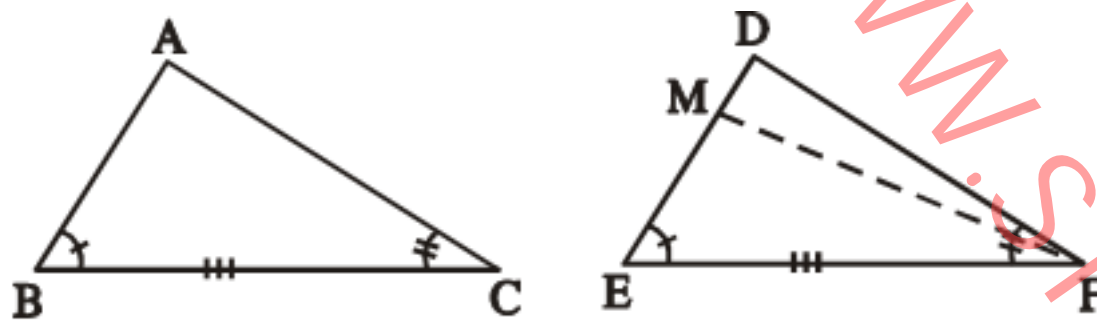
(iv) If $\Delta ABC \cong \Delta DEF$ and $\Delta ABC \cong \Delta PQR$, then $\Delta DEF \cong \Delta PQR$.

In any correspondence of two triangles, if two sides and their included angle of one triangle are congruent to the corresponding two sides and their included angle of the other, then the triangles are congruent.

In $\Delta ABC \longleftrightarrow \Delta DEF$, shown in the following figure,

If $\begin{cases} \overline{AB} \cong \overline{DE} \\ \angle A \cong \angle D \\ \overline{AC} \cong \overline{DF} \end{cases}$

then $\quad \Delta ABC \cong \Delta DEF \quad$ (S. A. S. Postulate)

Theorem 10.1.1

In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, then the triangles are congruent. (A.S.A. ≅ A.S.A.)

Given

In $\Delta ABC \longleftrightarrow \Delta DEF$

$\angle B \cong \angle E$, $\quad \overline{BC} \cong \overline{EF}$, $\quad \angle C \cong \angle F$.

To Prove

$\Delta ABC \cong \Delta DEF$

Construction

Suppose $\overline{AB} \ncong \overline{DE}$, take a point M on $\overline{DE}$ such that $\overline{AB} \cong \overline{ME}$. Join M to F

Proof

Statements	Reasons
In $\Delta ABC \longleftrightarrow \Delta MEF$	
$\overline{AB} \cong \overline{ME}$ ...... (i)	Construction
$\overline{BC} \cong \overline{EF}$ ...... (ii)	Given
$\angle B \cong \angle E$ ...... (iii)	Given
$\therefore \Delta ABC \cong \Delta MEF$	S.A.S. postulate
So, $\angle C \cong \angle MFE$	(Corresponding angles of congruent triangles)

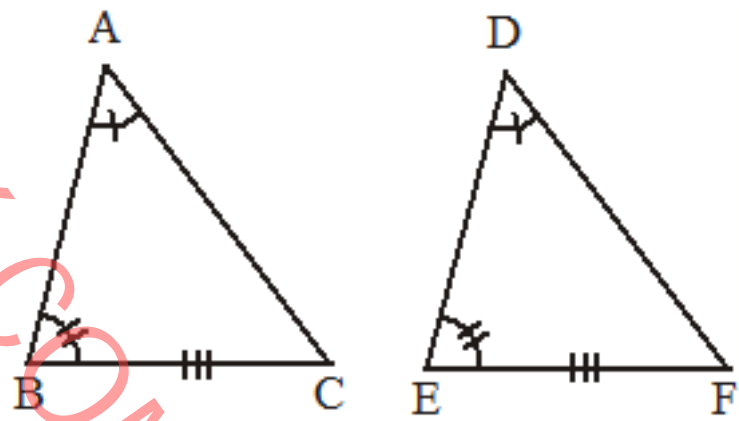
But, $\angle C \cong \angle DFE$	Given
$\therefore \angle DFE \cong \angle MFE$	Both congruent to $\angle C$
This is possible only if D and M are the same points, and $\overline{ME} \cong \overline{DE}$	
So, $\overline{AB} \cong \overline{DE}$ .... (iv)	$\overline{AB} \cong \overline{ME}$ (construction) and $\overline{ME} \cong \overline{DE}$ (proved)
Thus from (ii), (iii) and (iv), we have $\Delta ABC \cong \Delta DEF$	S.A.S. postulate

Corollary

In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, then the triangles are congruent. (S.A.A. ≅ S.A.A.)

Given

In $\Delta ABC \longleftrightarrow \Delta DEF$

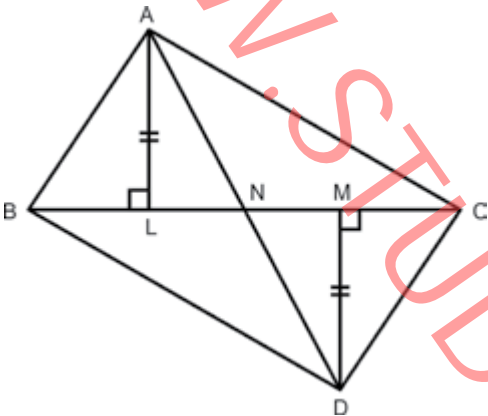
$\overline{BC} \cong \overline{EF}$, $\quad \angle A \cong \angle D$, $\quad \angle B \cong \angle E$

To Prove

$\Delta ABC \cong \Delta DEF$

Proof

Statements	Reasons
In $\Delta ABC \longleftrightarrow \Delta DEF$	
$\angle B \cong \angle E$	Given
$\overline{BC} \cong \overline{EF}$	Given
$\angle C \cong \angle F$	$\angle A \cong \angle D$, $\angle B \cong \angle E$, (Given)
$\therefore \Delta ABC \cong \Delta DEF$	A.S.A. $\cong$ A.S.A.

Example

If ΔABC and ΔDCB are on the opposite sides of common base $\overline{BC}$ such that
$\overline{AL} \perp \overline{BC}$, $\overline{DM} \perp \overline{BC}$ and
$\overline{AL} \cong \overline{DM}$, then $\overline{BC}$ bisects $\overline{AD}$.

Given

ΔABC and ΔDCB are on the opposite sides of $\overline{BC}$ such that $\overline{AL} \perp \overline{BC}$, $\overline{DM} \perp \overline{BC}$, $\overline{AL} \cong \overline{DM}$, and $\overline{AD}$ is cut by $\overline{BC}$ at N.

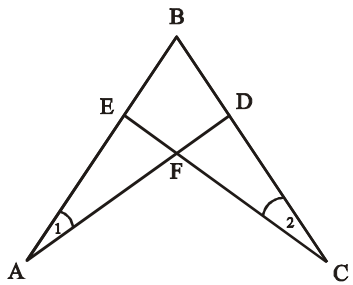
To Prove

$\overline{AN} \cong \overline{DN}$

Proof

Statements	Reasons
In $\Delta ALN \longleftrightarrow \Delta DMN$	
$\overline{AL} \cong \overline{DM}$	Given
$\angle ALN \cong \angle DMN$	Each angle is right angle
$\Delta ANL \cong \angle DNM$	Vertical angles
$\therefore \Delta ALN \cong \Delta DMN$	S.A.A. $\cong$ S.A.A.
Hence $\overline{AN} \cong \overline{DN}$	Corresponding sides of $\cong \Delta$s.

EXERCISE 10.1

1. In the given figure,
   $\overline{AB} \cong \overline{CB}$, $\angle 1 \cong \angle 2$.
   Prove that
   $\Delta ABD \cong \Delta CBE$.

2. From a point on the bisector of an angle, perpendiculars are drawn to the arms of the angle. Prove that these perpendiculars are equal in measure.
3. In a triangle ABC, the bisectors of $\angle B$ and $\angle C$ meet in a point I. Prove that I is equidistant from the three sides of ΔABC.

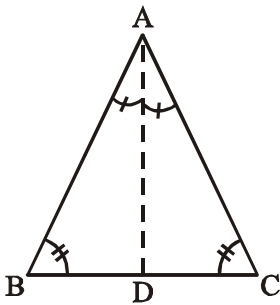
Theorem 10.1.2

If two angles of a triangle are congruent, then the sides opposite to them are also congruent.

Given

In ΔABC, $\angle B \cong \angle C$

To Prove

$\overline{AB} \cong \overline{AC}$

Construction

Draw the bisector of $\angle A$, meeting $\overline{BC}$ at the point D.

Proof

Statements	Reasons
In $\Delta ABD \longleftrightarrow \Delta ACD$	
$\overline{AD} \cong \overline{AD}$	Common
$\angle B \cong \angle C$	Given
$\angle BAD \cong \angle CAD$	Construction
$\therefore \Delta ABD \cong \Delta ACD$	S.A.A. $\cong$ S.A.A.
Hence $\overline{AB} \cong \overline{AC}$	(Corresponding sides of congruent triangles)

Example 1

If one angle of a right triangle *d* is of 30°, the hypotenuse is twice as long as the side opposite to the angle.

Given

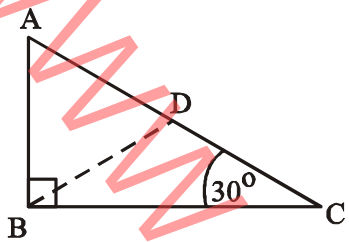
In ΔABC, $m\angle B = 90^0$ and

$m\angle C = 30^0$

To Prove

$m\overline{AC} = 2m\overline{AB}$

Construction

At B, construct $\angle CBD$ of 30°. Let $\overline{BD}$ cut $\overline{AC}$ at the point D.

Proof

Statements	Reasons
In ΔABD, $m\angle A = 60^0$	$m\angle ABC = 90°$, $m\angle C = 30°$
$m\angle ABD = m\angle ABC - m\angle CBD$	
$= 60^0$	$m\angle ABC = 90°$, $m\angle CBD = 30°$
$\therefore$ $m\angle ADB = 60^0$	Sum of measures of $\angle$s of a Δ is 180°
$\therefore$ ΔABD is equilateral	Each of its angles is equal to 60°
$\therefore$ $\overline{AB} \cong \overline{BD} \cong \overline{AD}$	Sides of equilateral Δ
In ΔBCD, $\overline{BD} \cong \overline{CD}$	$\angle C = \angle CBD$ (each of 30),
Thus $m\overline{AC} = m\overline{AD} + m\overline{CD}$	
$= m\overline{AB} + m\overline{AB}$	$\overline{AD} \cong \overline{AB}$ and $\overline{CD} \cong \overline{BD} \cong \overline{AB}$
$= 2(m\overline{AB})$	

Example 2

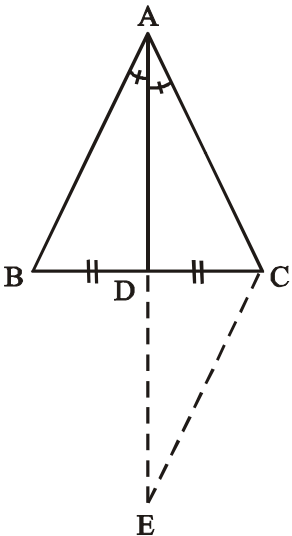
If the bisector of an angle of a triangle bisects the side opposite to it, the triangle is isosceles.

Given

In ΔABC, $\overline{AD}$ bisects $\angle A$ and $\overline{BD} \cong \overline{CD}$

$m\angle C = 30^0$

To Prove

$\overline{AB} \cong \overline{AC}$

Construction

Produce $\overline{AD}$ to E, and take $\overline{ED} \cong \overline{AD}$

Joint C to E.

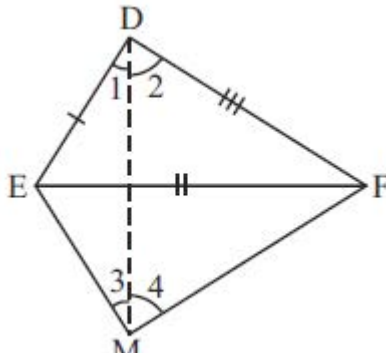
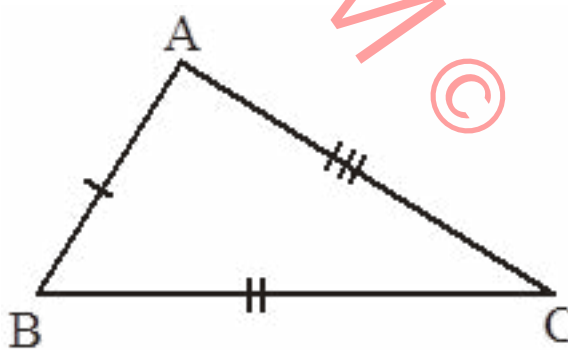
Proof

Statements	Reasons
In $\Delta ADB \longleftrightarrow \Delta EDC$	
$\overline{AD} \cong \overline{ED}$	Construction
$\angle ADB = \angle EDC$	Vertical angles
$\overline{BD} \cong \overline{CD}$	Given
$\therefore$ $\Delta ADB \cong \Delta EDC$	S.A.S. Postulate
$\therefore$ $\overline{AB} \cong \overline{EC}$ I	Corresponding sides of $\cong \Delta$s
and $\angle BAD \cong \angle E$	Corresponding angles of $\cong \Delta$s
But $\angle BAD \cong \angle CAD$	Given
$\therefore$ $\angle E \cong \angle CAD$	Each $\cong \angle BAD$
In ΔACE, $\overline{AC} \cong \overline{EC}$ II	$\angle E \cong \angle CAD$ (proved)
Hence $\overline{AB} \cong \overline{AC}$	From I and II

EXERCISE 10.2

1. Prove that any two medians of an equilateral triangle are equal in measure.
2. Prove that a point, which is equidistant from the end points of a line segment, is on the right bisector of the line segment.

Theorem 10.1.3

In a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, then the two triangles are congruent (S.S.S $\cong$ S.S.S).

Given

In $\Delta ABC \leftrightarrow \Delta DEF$

$\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ and $\overline{CA} \cong \overline{FD}$

To Prove

$\Delta ABC \cong \Delta DEF$

Construction

Suppose that in ΔDEF the side $\overline{EF}$ is not smaller than any of the remaining two sides. On $\overline{EF}$ costruct a ΔMEF in which, $\angle FEM \cong \angle B$ and $\overline{ME} \cong \overline{AB}$. Join D and M. As shown in the above figures we label some of the angles as 1, 2, 3 and 4.

Proof

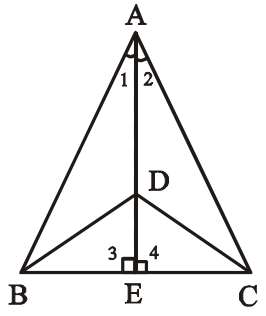
Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta MEF$	
$\overline{BC} \cong \overline{EF}$	Given
$\angle B = \angle FEM$	Construction
$\overline{AB} \cong \overline{ME}$	Construction
$\therefore$ $\Delta ABC \cong \Delta MEF$	S.A.S postulate
and $\overline{CA} \cong \overline{FM}$ .... (i)	(corresponding sides of congruent triangles)
Also $\overline{CA} \cong \overline{FD}$ .... (ii)	Given
$\therefore$ $\overline{FM} \cong \overline{FD}$	{From (i) and (ii)}
In ΔFDM	
$\angle 2 \cong \angle 4$ .... (iii)	$\overline{FM} \cong \overline{FD}$ (proved)
Similarly $\angle 1 \cong \angle 3$ .... (iv)	
$\therefore$ $m\angle 2 + m\angle 1 = m\angle 4 + m\angle 3$	{from (iii) and (iv)}
$\therefore$ $m\angle EDF = m\angle EMF$	
Now, in $\Delta ADB \leftrightarrow \Delta EDC$	
$\overline{FD} \cong \overline{FM}$	Proved
and $m\angle EDF \cong m\angle EMF$	Proved
$\overline{DE} \cong \overline{ME}$	Each one $\cong \overline{AB}$
$\therefore$ $\Delta DEF \cong \Delta MEF$	S.A.S. postulate
Also $\Delta ABC \cong \Delta MEF$	Proved
Hence $\Delta ABC \cong \Delta DEF$	Each $\Delta \cong \Delta MEF$ (Proved)

Corollary

If two isosceles triangles are formed on the same side of their common base, the line through their vertices would be the right bisector of their common base.

Given

ΔABC and ΔDBC are formed on the same side of $\overline{BC}$ such that

$\overline{AB} \cong \overline{AC}$, $\overline{DB} \cong \overline{DC}$, $\overline{AD}$ meets $\overline{BC}$ at E.

To Prove

$\overline{BE} \cong \overline{CE}$, $\overline{AE} \perp \overline{BC}$

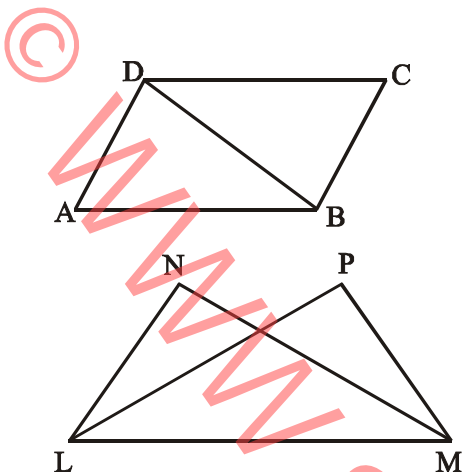
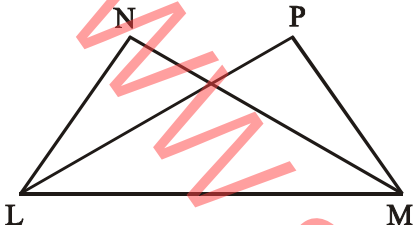
Proof

Statements	Reasons
In $\Delta ADB \longleftrightarrow \Delta ADC$	
$\overline{AB} \cong \overline{AC}$	Given
$\overline{DB} \cong \overline{DC}$	Given
$\overline{AD} \cong \overline{AD}$	Common
$\therefore$ $\Delta ADB \cong \Delta ADC$	S.S.S $\cong$ S.S.S.
$\therefore$ $\angle 1 \cong \angle 2$	Corresponding angles of $\cong \Delta$s
In $\Delta ABE \longleftrightarrow \Delta ACE$	
Also $\overline{AB} \cong \overline{AC}$	Given
$\therefore$ $\angle 1 \cong \angle 2$	Proved
$\overline{AE} \cong \overline{AE}$	Common
$\therefore$ $\Delta ABE \cong \Delta ACE$	S.A.S. postulate
$\therefore$ $\overline{BE} \cong \overline{CE}$	Corresponding sides of $\cong \Delta$s
$\angle 3 \cong \angle 4$ I	Corresponding sides of $\cong \Delta$s
$m\angle 3 + m\angle 4 = 180^0$ II	Supplementary angles Postulate
$\therefore$ $m\angle 3 = m\angle 4 = 90^0$	From I and II
Hence $\overline{AE} \perp \overline{BC}$	

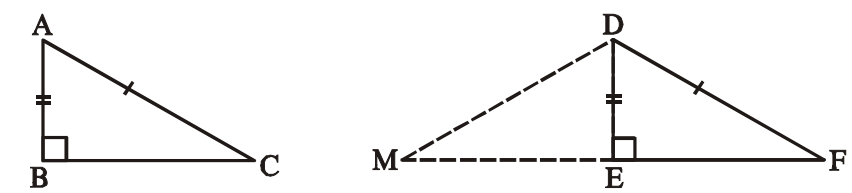
Corollary:

An equilateral triangle is an equiangular triangle.

EXERCISE 10.3

1. In the figure, $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$.
   Prove that $\angle A \cong \angle C$, $\angle ABC \cong \angle ADC$.

2. In the figure, $\overline{LN} \cong \overline{MP}$, $\overline{MN} \cong \overline{LP}$.
   Prove that $\angle N \cong \angle P$, $\angle NML \cong \angle PLM$.

3. Prove that the median bisecting the base of an isosceles triangle bisects the vertex angle and it is prependicular to the base.

Theorem 10.1.4

If in the correspondence of the two right-angled triangles, the hypotenuse and one side of one triangle are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent. (H.S ≅ H.S).

Given

In $\Delta ABC \longleftrightarrow \Delta DEF$

$\angle B \cong \angle E$ (right angles)

$\overline{CA} \cong \overline{FD}$, $\overline{AB} \cong \overline{DE}$

To Prove

$\Delta ABC \cong \Delta DEF$

Construction

Produce $\overline{FE}$ to a point M such that $\overline{EM} \cong \overline{BC}$ and join the points D and M.

Proof

Statements	Reasons
$m\angle DEF + m\angle DEM = 180^\circ$ ......(i)	(Supplementary angles)
Now $m\angle DEF = 90^\circ$ .....(ii)	Given
$\therefore$ $m\angle DEM = 90^\circ$	{from (i) and (ii)}
In $\Delta ABC \leftrightarrow \Delta DEM$	
$\overline{BC} \cong \overline{EM}$	(construction)
$\angle ABC \cong \angle DEM$	(each $\angle$ equal to 90°)
$\overline{AB} \cong \overline{DE}$	(given)
$\therefore$ $\Delta ABC \cong \Delta DEM$	S.A.S. postulate
and $\angle C \cong \angle M$	(Corresponding angles of congruent triangles)
$\overline{CA} \cong \overline{MD}$	(Corresponding sides of congruent triangles)
But $\overline{CA} \cong \overline{FD}$	(given)
$\therefore$ $\overline{MD} \cong \overline{FD}$	each is congruent to $\overline{CA}$
In ΔDMF	
$\angle F \cong \angle M$	$\overline{MD} \cong \overline{FD}$ (proved)
But $\angle C \cong \angle M$	(proved)
$\angle C \cong \angle F$	(each is congruent to $\angle M$)
In $\Delta ABC \leftrightarrow \Delta DEF$	
$\overline{AB} \cong \overline{DE}$	(given)
$\angle ABC \cong \angle DEF$	(given)
$\angle C \cong \angle F$	(proved)
Hence $\Delta ABC \cong \Delta DEF$	(S.A.A. ≅ S.A.A)

Example

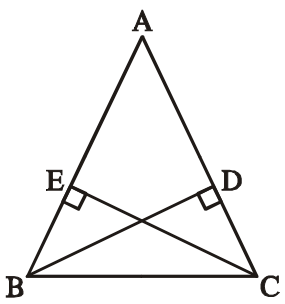
If perpendiculars from two vertices of a triangle to the opposite sides are congruent, then the triangle is isosceles.

Given

In ΔABC, $\overline{BD} \perp \overline{AC}$, $\overline{CE} \perp \overline{AB}$

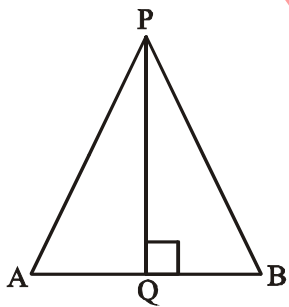
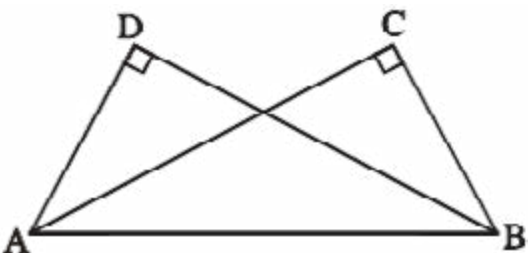
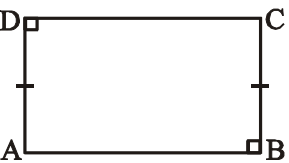
Such that $\overline{BD} \cong \overline{CE}$

To Prove

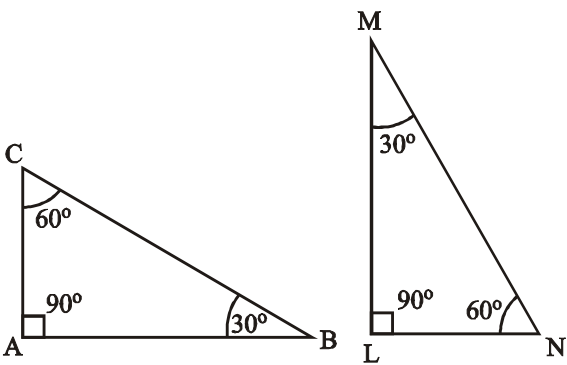
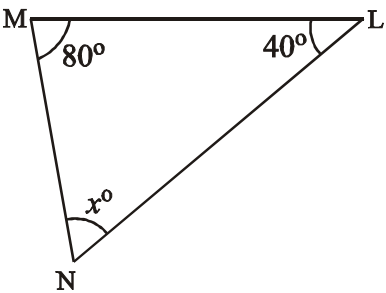
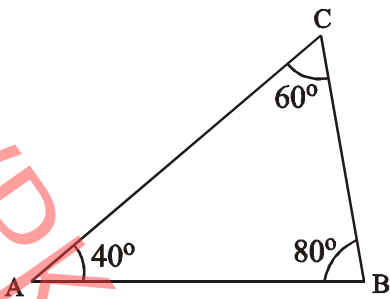
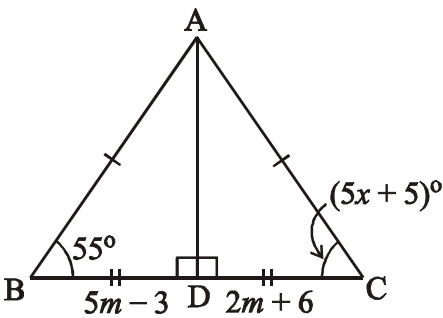
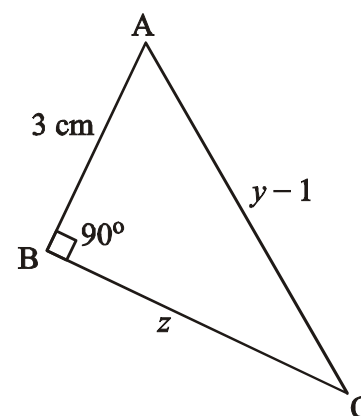
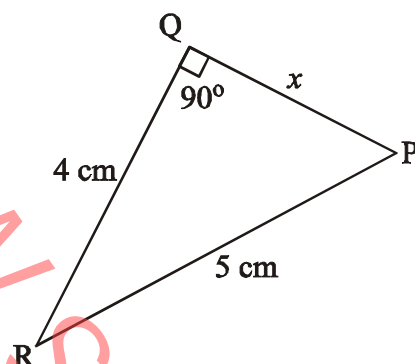
$\overline{AB} \cong \overline{AC}$

Proof

Statements	Reasons
In $\Delta BCD \longleftrightarrow \Delta CBE$	
$\angle BCD \cong \angle BEC$	$\overline{BD} \perp \overline{AC}$, $\overline{CE} \perp \overline{AB}$ (given) $\Rightarrow$ each angle $= 90^0$
$\overline{BC} \cong \overline{BC}$	Common hypotenuse
$\overline{BD} \cong \overline{CE}$	Given
$\therefore$ $\Delta BCD \cong \Delta CBE$	H.S. $\cong$ H.S.
$\therefore$ $\angle BCD \cong \angle CBE$	Corresponding angles Δs
Thus $\angle BCD \cong \angle CBE$	
Hence $\overline{AB} \cong \overline{AC}$	In ΔABC, $\angle BCA \cong \angle CBA$

EXERCISE 10.4

1. In ΔPAB of figure, $\overline{PQ} \perp \overline{AB}$ and $\overline{PA} \cong \overline{PB}$, proved that $\overline{AQ} \cong \overline{BQ}$ and $\angle APQ \cong \angle BPQ$.

2. In the figure, $m\angle C = m\angle D = 90^0$ and $\overline{BC} \cong \overline{AD}$. Prove that $\overline{AC} \cong \overline{BD}$, and $\angle BAC \cong \angle ABD$.

3. In the figure, $m\angle B = m\angle D = 90^0$ and $\overline{AD} \cong \overline{BC}$. Prove that ABCD is a rectangle.

REVIEW EXERCISE 10

1. Which of the following are true and which are false?
   (i) A ray has two end points. ......
   (ii) In a triangle, there can be only one right angle. ......
   (iii) Three points are said to be collinear, if they lie on same line. ...
   (iv) Two parallel lines intersect at a point. ......
   (v) Two lines can intersect only at one point. ......
   (vi) A triangle of congruent sides has non-congruent angles. ......

2. If $\Delta ABC \cong \Delta LMN$, then
   (i) $m\angle M \cong$ ....
   (ii) $m\angle N \cong$ ....
   (iii) $m\angle A \cong$ ....

3. If $\Delta ABC \cong \Delta LMN$, then find the unknown x.

4. Find the value of unknowns for the given congruent triangles.

5. If PQR $\cong$ ABC, then find the unknowns.

SUMMARY

In this unit we stated and proved the following theorems:

- In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, the two triangles are congruent. (A.S.A $\cong$ A.S.A.)
- If two angles of a triangle are congruent, then the sides opposite to them are also congruent.
- In a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, then the two triangles are congruent (S.S.S $\cong$ S.S.S).
- If in the correspondence of the two right-angled triangles, the hypotenuse and one side of one triangle are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent. (H.S $\cong$ H.S).
- Two triangles are said to be congruent, if there exists a correspondence between them such that all the corresponding sides and angles are congruent.

CHAPTER

11

PARALLELOGRAMS AND TRIANGLES

Animation 11.1: Triangle to Square
Source & Credit: takayaiwamoto

Students Learning Outcomes

After studying this unit, the students will be able to:

- prove that in a parallelogram
  - the opposite sides are congruent,
  - the opposite angles are congruent,
  - the diagonals bisect each other.
- prove that if two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.
- prove that the line segment, joining the midpoints of two sides of a triangle, is parallel to the third side and is equal to one half of its length.
- prove that the medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.
- prove that if three or more parallel lines make congruent segments on a transversal, they also intercept congruent segments on any other line that cuts them.

Introduction

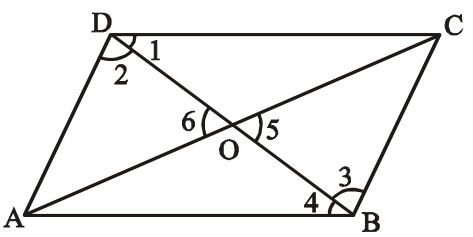
Before proceeding to prove the theorems in this unit the students are advised to recall definitions of polygons like parallelogram, rectangle, square, rhombus, trapezium etc. and in particular triangles and their congruency.

Theorem 11.1.1

In a parallelogram

(i) Opposite sides are congruent.

(ii) Opposite angles are congruent.

(iii) The diagonals bisect each other.

Given

In a quadrilateral ABCD, $\overline{AB} \parallel \overline{DC}$, $\overline{BC} \parallel \overline{AD}$ and the diagonals $\overline{AC}$, $\overline{BD}$ meet each other at point O.

To Prove

(i) $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$

(ii) $\angle ADC \cong \angle ABC$, $\angle BAD \cong \angle BCD$

(iii) $\overline{OA} \cong \overline{OC}$, $\overline{OB} \cong \overline{OD}$

Construction

In the figure as shown, we label the angles as $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$, and $\angle 6$

Proof

Statements	Reasons
In $\Delta ABD \longleftrightarrow \Delta CDB$	
$\angle 4 \cong \angle 1$	alternate angles
$\overline{BD} \cong \overline{BD}$	Common
$\angle 2 \cong \angle 3$	alternate angles
$\therefore \Delta ABD \cong \Delta CDB$	A.S.A. ≅ A.S.A.
So, $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$	(corresponding sides of congruent triangles)
and $\angle A \cong \angle C$	(corresponding angles of congruent triangles)
(ii) Since	
$\angle 1 \cong \angle 4$ ...... (a)	Proved
and $\angle 2 \cong \angle 3$ ...... (b)	Proved
$\therefore m\angle 1 + m\angle 2 = m\angle 4 + m\angle 3$	from (a) and (b)
or $m\angle ADC \cong m\angle ABC$	Proved in (i)
$\angle ADC \cong \angle ABC$	
and	
$\angle BAD \cong \angle BCD$	
(iii) In $\Delta BOC \longleftrightarrow \Delta DOA$	
$\overline{BC} \cong \overline{AD}$	Proved in (i)
$\angle 5 \cong \angle 6$	vertical angles
$\angle 3 \cong \angle 2$	Proved
$\therefore \Delta BOC \cong \Delta DOA$	(A.A.S. ≅ A. A. S.)
Hence $\overline{OC} \cong \overline{OA}$, $\overline{OB} \cong \overline{OD}$	(corresponding sides of congruent triangles)

Corollary

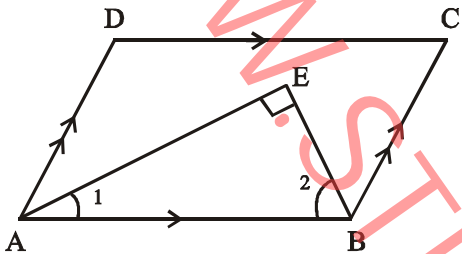
Each diagonal of a parallelogram bisects it into two congruent triangles.

Example

The bisectors of two angles on the same side of a parallelogram cut each other at right angles.

Given

A parallelogram ABCD, in which $\overline{AB} \parallel \overline{DC}$, $\overline{AD} \parallel \overline{BC}$.

The bisectors of $\angle A$ and $\angle B$ cut each other at E.

To Prove

$m\angle E = 90^0$

Construction

Name the angles $\angle 1$ and $\angle 2$ as shown in the figure.

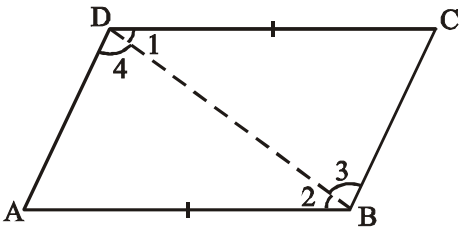
Proof

Statements	Reasons
$m\angle 1 + m\angle 2$	
$= \frac{1}{2}(m\angle BAD + m\angle ABC)$	$\left\{ \begin{array}{l} m\angle 1 = m\frac{1}{2}\angle BAD, \\ m\angle 2 = m\frac{1}{2}\angle ABC \end{array} \right.$
$= \frac{1}{2}(180^0)$	{Int. angles on the same side of $\overline{AB}$ which cuts $\parallel$ segments $\overline{AD}$ and $\overline{BC}$ are supplementary.
$=90^0$	
Hence in ΔABE, $m\angle E = 90^0$	$m\angle 1 + m\angle 2 = 90^0$ (Proved)

EXERCISE 11.1

1. One angle of a parallelogram is 130^0. Find the measures of its remaining angles.
2. One exterior angle formed on producing one side of a parallelogram is 40^0. Find the measures of its interior angles.

Theorem 11.1.2

If two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.

Given

In a quadrilateral ABCD,
$\overline{AB} \cong \overline{DC}$ and $\overline{AB} \parallel \overline{DC}$

To Prove

ABCD is a parallelogram.

Construction

Join the point B to D and in the figure, name the angles as indicated: $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$

Proof

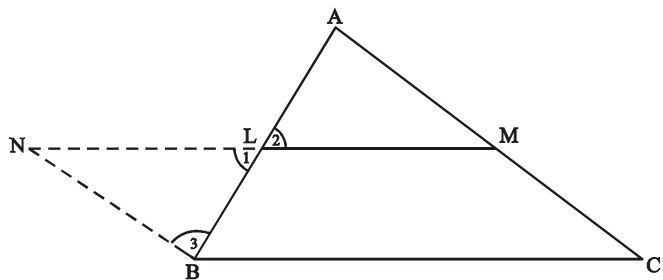
Statements	Reasons
In $\Delta ABD \longleftrightarrow \Delta CDB$	
$\overline{AB} \cong \overline{DC}$,	given
$\angle 2 \cong \angle 1$	alternate angles
$\overline{BD} \cong \overline{BD}$	Common
$\therefore$ $\Delta ABD \cong \Delta CDB$	S.A.S. postulate
Now $\angle 4 \cong \angle 3$ ....(i)	(corresponding angles of congruent triangles)
$\therefore$ $\overline{AD} \parallel \overline{BC}$ ....(ii)	from (i)
and $\overline{AD} = \overline{BC}$ ....(iii)	corresponding sides of congruent $\triangle$s

Also $\overline{AB} \parallel \overline{DC}$ ....(iv)	given
Hence ABCD is a parallelogram	from (ii) – (iv)

EXERCISE 11.2

1. Prove that a quadrilateral is a parallelogram if its
   (a) opposite angles are congruent. (b) diagonals bisect each other.
2. Prove that a quadrilateral is a parallelogram if its opposite sides are congruent.

Theorem 11.1.3

The line segment, joining the mid-points of two sides of a triangle, is parallel to the third side and is equal to one half of its length.

Given

In ΔABC, The mid-points of $\overline{AB}$ and $\overline{AC}$ are L and M respectively.

To Prove

$\overline{LM} \parallel \overline{BC}$ and $m\overline{LM} = \frac{1}{2} m\overline{BC}$

Construction

Join M to L and produce $\overline{ML}$ to N such that $\overline{ML} \cong \overline{LN}$.
Join N to B and in the figure, name the angles as $\angle 1, \angle 2$ and $\angle 3$ as shown.

Proof

Statements	Reasons
In $\Delta BLN \longleftrightarrow \Delta ALM$	
$\overline{BL} \cong \overline{AL}$	Given
$\angle 1 \cong \angle 2$	vertical angles
$\overline{NL} \cong \overline{ML}$	Construction
$\therefore$ $\Delta BLN \cong \Delta ALM$	S.A.S. postulate
$\therefore$ $\angle A \cong \angle 3$ ....(i)	(corresponding angles of congruent triangles)
and $\overline{NB} \cong \overline{AM}$ ....(ii)	(corresponding sides of congruent triangles)
But $\overline{NB} \parallel \overline{AM}$	From (i), alternate $\angle$s
Thus $\overline{NB} \parallel \overline{MC}$ ....(iii)	(M is a point of $\overline{AC}$)
$\overline{MC} \cong \overline{AM}$ ....(iv)	Given
$\overline{NB} \cong \overline{MC}$ ....(v)	{from (ii) and (iv)}
$\therefore$ BCMN is a parallelogram	from (iii) and (v)
$\therefore$ $\overline{BC} \parallel \overline{LM}$ or $\overline{BC} \parallel \overline{NL}$	(opposite sides of a parallelogram BCMN)
$\overline{BC} \cong \overline{NM}$ ....(vi)	(opposite sides of a parallelogram)
$m\overline{LM} = m\frac{1}{2}\overline{NM}$ ....(vii)	Construction
Hence $m\overline{LM} = \frac{1}{2} m\overline{BC}$	{from (vi) and (vii)}

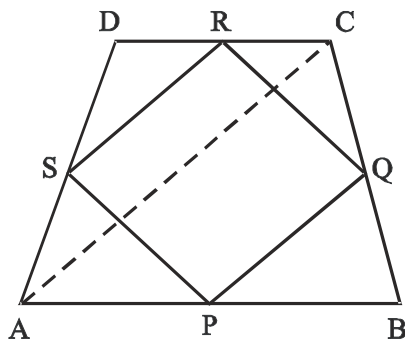
Note that instead of producing $\overline{ML}$ to N, we can take N on $\overline{LM}$ produced.

Example

The line segments, joining the mid-points of the sides of a quadrilateral, taken in order, form a parallelogram.

Given

A quadrilateral ABCD, in which P is the mid-point of $\overline{AB}$, Q is the mid-point of $\overline{BC}$, R is the mid-point of $\overline{CD}$, S is the mid-point of $\overline{DA}$.

P is joined to Q, Q is joined to R.
R is joined to S and S is joined to P.

To Prove

PQRS is a parallelogram.

Construction

Join A to C.

Proof

Statements	Reasons
In ΔDAC, $\overline{SR} \parallel \overline{AC}$ $m\overline{SR} = m\frac{1}{2}\overline{AC}$	S is the midpoint of $\overline{DA}$ R is the midpoint of $\overline{CD}$
In ΔBAC, $\overline{PQ} \parallel \overline{AC}$ $m\overline{PQ} = m\frac{1}{2}\overline{AC}$	P is the midpoint of $\overline{AB}$ Q is the mid-point of $\overline{BC}$
$\overline{SR} \parallel \overline{PQ}$	Each $\parallel \overline{AC}$
$m\overline{SR} = m\overline{PQ}$	Each $= m\frac{1}{2}\overline{AC}$
Thus PQRS is a parallelogram	$\overline{SR} \parallel \overline{PQ}$, $m\overline{SR} = m\overline{PQ}$ (proved)

EXERCISE 11.3

1. Prove that the line-segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.
2. Prove that the line-segments joining the midpoints of the opposite sides of a rectangle are the right-bisectors of each other.
   [**Hint**: Diagonals of a rectangle are congruent.]
3. Prove that the line-segment passing through the midpoint of one side and parallel to another side of a triangle also bisects the third side.

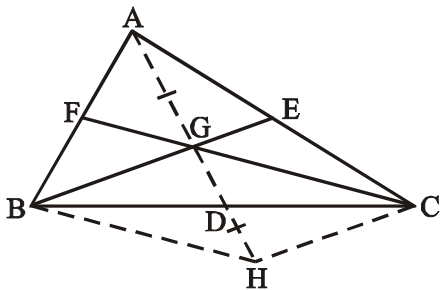
Theorem 11.1.4

The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.

Given

$\triangle ABC$

To Prove

The medians of the $\triangle ABC$ are concurrent and the point of concurrency is the point of trisection of each median.

Construction

Draw two medians $\overline{BE}$ and $\overline{CF}$ of the $\triangle ABC$ which intersect each other at G. Join A to G and produce it to point H such that $AG \cong \overline{GH}$. Join H to the points B and C.

$\overline{AH}$ intersects $\overline{BC}$ at the point D.

Proof

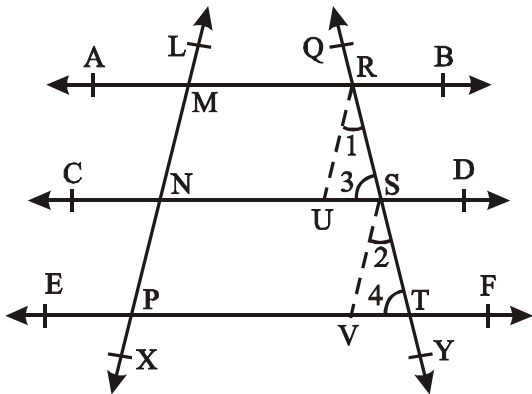
Statements	Reasons
In $\triangle ACH$, $\overline{GE} \parallel \overline{HC}$	G and E are mid-points of sides AH and AC respectively
or $\overline{BE} \parallel \overline{HC}$ ....(i)	G is a point of BE
Similarly, $\overline{CF} \parallel \overline{HB}$ ....(ii)	
$\therefore$ BHCG is a parallelogram	from (i) and (ii)
and $m\overline{GD} = m\overline{GH}$ ....(iii)	(diagonals $\overline{BC}$ and $\overline{GH}$ of a parallelogram BHCG intersect each other at point D)
$\overline{BD} \cong \overline{CD}$ $\overline{AD}$ is a meadian of $\triangle ABC$	
Meadians $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ pass through the point G	(G is the intersecting point of $\overline{BE}$ and $\overline{CF}$ and $\overline{AD}$ pass through it.)
Now $\overline{GH} \cong \overline{AG}$ ....(iv)	construction

∴ $m\overline{GD} = \frac{1}{2} m\overline{AG}$ and G is the point of trisection of $\overline{AD}$ ....(v) Similarly it can be proved that G is also the point of trisection of $\overline{CF}$ and $\overline{BE}$	from (iii) and (iv)

EXERCISE 11.4

1. The distances of the points of concurrency of the median of a triangle from its vertices are respectively 1.2 cm, 1.4 cm and 1.6 cm. Find the lengths of its medians.
2. Prove that the point of concurrency of the medians of a triangle and the triangle which is made by joining the mid-point of its sides is the same.

Theorem 11.1.5

If three or more parallel lines make congruent segments on a transversal, they also intercept congruent segments on any other line that cuts them.

Given

$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$

The transversal LX intersects $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ at the points M, N and P respectively, such that $\overline{MN} \cong \overline{NP}$. The transversal $\overleftrightarrow{QY}$ intersects them at points R, S and T respectively.

To Prove

$\overline{RS} \cong \overline{ST}$

Construction

From R, draw $\overline{RU} \parallel \overline{LX}$, which meets $\overline{CD}$ at U. From S, draw $\overline{SV} \parallel \overline{LX}$ which meets $\overline{EF}$ at V. As shown in the figure let the angles be labelled as

$\angle 1, \angle 2, \angle 3$ and $\angle 4$

Proof

Statements	Reasons
MNUR is a parallelogram	$\overline{RU} \parallel \overline{LX}$ (construction) $\overline{AB} \parallel \overline{CD}$ (given)
∴ $\overline{MN} \cong \overline{RU}$ (i)	(opposite sides of a parallelogram)
Similarly, $\overline{NP} \cong \overline{SV}$ (ii)	
But $\overline{MN} \cong \overline{NP}$ (iii)	Given
∴ $\overline{RU} \cong \overline{SV}$	{from (i), (ii) and (iii)}
Also $\overline{RU} \parallel \overline{SV}$	each $\parallel \overleftrightarrow{LX}$ (construction)
∴ $\angle 1 \cong \angle 2$	Corresponding angles
and $\angle 3 \cong \angle 4$	Corresponding angles
In $\Delta RUS \longleftrightarrow \Delta SVT$,	
$\overline{RU} \cong \overline{SV}$	Proved
$\angle 1 \cong \angle 2$	Proved
$\angle 3 \cong \angle 4$	Proved
∴ $\Delta RUS \cong \Delta SVT$	S.A.A. ≅ S.A.A.
Hence $\overline{RS} \cong \overline{ST}$	(corresponding sides of congruent triangles)

Note: This theorem helps us in dividing line segment into parts of equal lengths. It is also used in the division of a line segment into proportional parts.

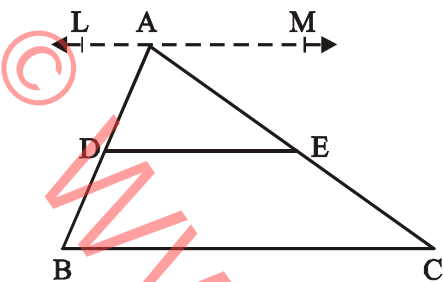
Corollaries

(i) A line, through the mid-point of one side, parallel to another side of a triangle, bisects the third side.

Given

In $\triangle ABC$, D is the mid-point of $\overline{AB}$.
$\overline{DE} \parallel \overline{BC}$ which cuts $\overline{AC}$ at E.

To Prove

$\overline{AE} \cong \overline{EC}$

Construction

Through A, draw $\overleftrightarrow{LM} \parallel \overline{BC}$.

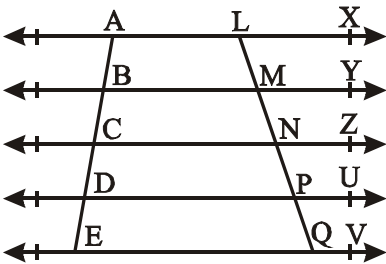
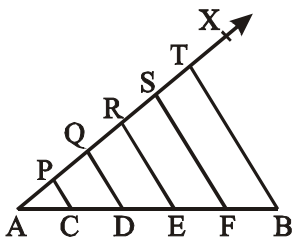
Proof

Statements	Reasons
Intercepts cut by $\overleftrightarrow{LM}$, $\overline{DE}$, $\overline{BC}$ on $\overline{AC}$ are congruent. i.e., $\overline{AE} \cong \overline{EC}$.	Intercepts cut by parallels $\overleftrightarrow{LM}$, $\overline{DE}$, $\overline{BC}$ on $\overline{AB}$ are congruent (given)

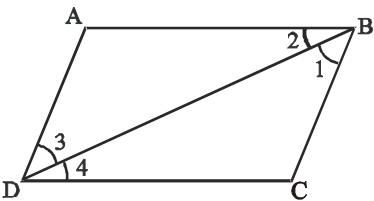
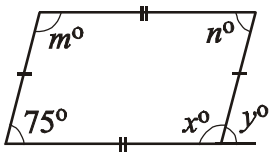
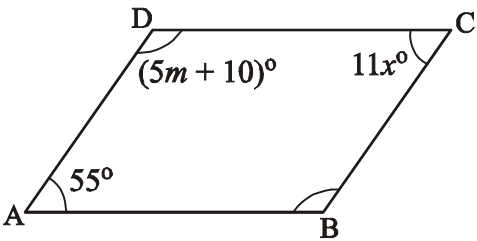
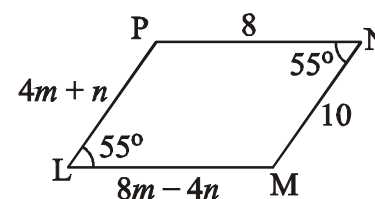
(ii) The parallel line from the mid-point of one non-parallel side of a trapezium to the parallel sides bisects the other non-parallel side.

(iii) If one side of a triangle is divided into congruent segments, the line drawn from the point of division parallel to the other side will make congruent segments on third side.

EXERCISE 11.5

1. In the given figure
$\overleftrightarrow{AX} \parallel \overleftrightarrow{BY} \parallel \overleftrightarrow{CZ} \parallel \overleftrightarrow{DU} \parallel \overleftrightarrow{EV}$
and $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DE}$
If $m\overline{MN}$ = 1cm, then find the length of $\overline{LN}$ and $\overline{LQ}$.

2. Take a line segment of length 5.5 cm and divide it into five congruent parts.
[Hint: Draw an acute angle $\angle BAX$ on $\overline{AX}$ take $\overline{AP} \cong \overline{PQ} \cong \overline{QR} \cong \overline{RS} \cong ST$.
Join T to B. Draw lines parallel to $\overline{TB}$ from the points P, Q, R and S.]

REVIEW EXERCISE 11

1. Fill in the blanks.
   (i) In a parallelogram opposite sides are ....
   (ii) In a parallelogram opposite angles are ....
   (iii) Diagonals of a parallelogram .... each other at a point.
   (iv) Medians of a triangle are ....
   (v) Diagonal of a parallelogram divides the parallelogram into two .... triangles

2. In parallelogram ABCD
   (i) $m\overline{AB}$.......$m\overline{DC}$ (ii) $m\overline{BC}$........$m\overline{AD}$
   (iii) $m\angle 1 \cong$......... (iv) $m\angle 2 \cong$...........

3. Find the unknowns in the given figure.

4. If the given figure ABCD is a parallelogram, then find x, m.

5. The given figure LMNP is a parallelogram. Find the value of m, n.

6. In the question 5, sum of the opposite angles of the parallelogram is 110^0, find the remaining angles.

SUMMARY

In this unit we discussed the following theorems and used them to solve some exercises. They are supplemented by unsolved exercises to enhance applicative skills of the students.

- In a parallelogram
  (i) Opposite sides are congruent.
  (ii) Opposite angles are congruent.
  (iii) The diagonals bisect each other.
- If two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.
- The line segment, joining the mid-points of two sides of a triangle, is parallel to the third side and is equal to one half of its length.
- The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.
- If three or more parallel lines make congruent segments on a transversal, they also intercept congruent segments on any other line that cuts them.

CHAPTER

12

LINE BISECTORS AND ANGLE BISECTORS

Animation 12.1: Angle- Bisectors
Source & Credit: mathsonline

Students Learning Outcomes

After studying this unit, the students will be able to:

- Prove that any point on the right bisector of a line segment is equidistant from its end points.
- Prove that any point equidistant from the end points of a line segment is on the right bisector of it.
- Prove that the right bisectors of the sides of a triangle are concurrent.
- Prove that any point on the bisector of an angle is equidistant from its arms.
- Prove that any point inside an angle, equidistant from its arms, is on the bisector of it.
- Prove that the bisectors of the angles of a triangle are concurrent.

Introduction

In this unit, we will prove theorems and their converses, if any, about right bisector of a line segment and bisector of an angle. But before that it will be useful to recall the following definitions:

Right Bisector of a Line Segment

A line is called a right bisector of a line segment if it is perpendicular to the line segment and passes through its midpoint.

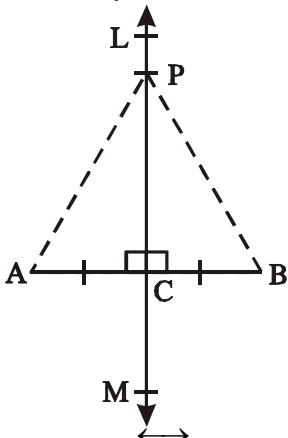
Bisector of an Angle

A ray BP is called the bisector of $\angle ABC$, if P is a point in the interior of the angle and $m\angle ABP = m\angle PBC$.

Theorem 12.1.1

Any point on the right bisector of a line segment is equidistant from its end points.

Given

A line LM intersects the line segment AB at the point C. Such that $\overleftrightarrow{LM} \perp \overline{AB}$ and $\overline{AC} \cong \overline{BC}$. P is a point on $\overleftrightarrow{LM}$.

To Prove

$\overline{PA} \cong \overline{PB}$

Construction

Join P to the points A and B.

Proof

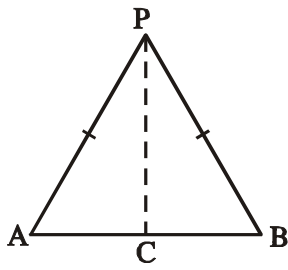
Statements	Reasons
In $\Delta ACP \longleftrightarrow \Delta BCP$	
$\overline{AC} \cong \overline{BC}$	given
$\angle ACP \cong \angle BCP$	given $\overline{PC} \perp \overline{AB}$, so that each $\angle$ at C = 90^0
$\overline{PC} \cong \overline{PC}$	Common
$\Delta ACP \cong \Delta BCP$	S.A.S. postulate
Hence $\overline{PA} \cong \overline{PB}$	(corresponding sides of congruent triangles)

Theorem 12.1.2

{Converse of Theorem 12.1.1}

Any point equidistant from the end points of a line segment is on the right bisector of it.

Given

$\overline{AB}$ is a line segment. Point P is such that $\overline{PA} \cong \overline{PB}$.

To Prove

The point P is on the right bisector of $\overline{AB}$.

Construction

Joint P to C, the mid-point of $\overline{AB}$.

Proof

Statements	Reasons
In $\Delta ACP \longleftrightarrow \Delta BCP$	
$\overline{PA} \cong \overline{PB}$	given
$\overline{PC} \cong \overline{PC}$	Common

$\overline{AC} \cong \overline{BC}$	Construction
$\therefore$ $\Delta ACP \cong \Delta BCP$	S.S.S. $\cong$ S.S.S.
$\angle ACP \cong \angle BCP$ ......(i)	(corresponding angles of congruent triangles)
But $m\angle ACP + m\angle BCP = 180°$ ..... (ii)	Supplementary angles
$\therefore$ $m\angle ACP = m\angle BCP = 90°$	from (i) and (ii)
i.e., $\overline{PC} \perp \overline{AB}$ ...... (iii)	$m\angle ACP = 90°$ (proved)
Also $\overline{CA} \cong \overline{CB}$ ...... (iv)	construction
$\therefore$ $\overline{PC}$ is a right bisector of $\overline{AB}$.	from (iii) and (iv)
i.e., the point P is on the right bisector of $\overline{AB}$.	

EXERCISE 12.1

1. Prove that the centre of a circle is on the right bisectors of each of its chords.
2. Where will be the centre of a circle passing through three non-collinear points? And why?
3. Three villages P, Q and R are not on the same line. The people of these villages want to make a Children Park at such a place which is equidistant from these three villages. After fixing the place of Children Park, prove that the Park is equidistant from the three villages.

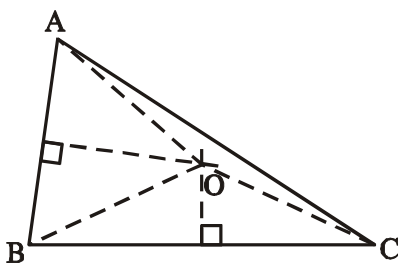
Theorem 12.1.3

The right bisectors of the sides of a triangle are concurrent.

Given

ΔABC

To Prove

The right bisectors of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ are concurrent.

Construction

Draw the right bisectors of $\overline{AB}$ and $\overline{BC}$ which meet each other at the point O. Join O to A, B and C.

Proof

Statements	Reasons
In $\overline{OA} \cong \overline{OB}$ ...... (i)	(Each point on right bisector of a segment is equidistant from its end points)
$\overline{OB} \cong \overline{OC}$ ...... (ii)	as in (i)
$\overline{OA} \cong \overline{OC}$ ...... (iii)	from (i) and (ii)
$\therefore$ Point O is on the right bisector of $\overline{CA}$. ...... (iv)	(O is equidistant from A and C)
But point O is on the right bisector of $\overline{AB}$ and of $\overline{BC}$ ...... (v)	construction
Hence the right bisectors of the three sides of a triangle are concurrent at O.	{from (iv) and (v)}

Observe that

(a) The right bisectors of the sides of an acute triangle intersect each other inside the triangle.

(b) The right bisectors of the sides of a right triangle intersect each other on the hypotenuse.

(c) The right bisectors of the sides of an obtuse triangle intersect each other outside the triangle.

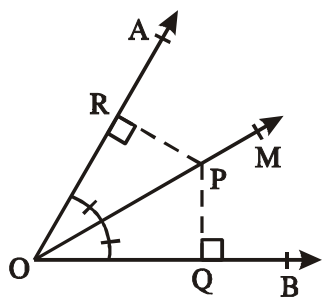
Theorem 12.1.4

Any point on the bisector of an angle is equidistant from its arms.

Given

A point P is on $\overrightarrow{OM}$, the bisector of $\angle AOB$.

To Prove

$\overline{PQ} \cong \overline{PR}$ i.e., P is equidistant from $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Construction

Draw $\overline{PR} \perp \overrightarrow{OA}$ and $\overline{PQ} \perp \overrightarrow{OB}$

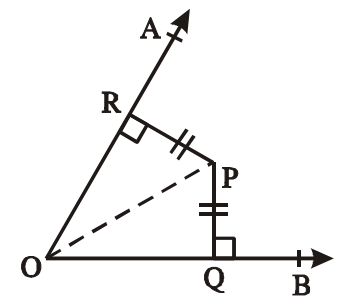
Proof

Statements	Reasons
In $\Delta POQ \longleftrightarrow \Delta POR$	
$\overline{OP} \cong \overline{OP}$	common
$\angle PQO \cong \angle PRO$	construction
$\angle POQ \cong \angle POR$	given
$\therefore$ $\Delta POQ \cong \Delta POR$	S.A.A. $\cong$ S.A.A.
Hence $\overline{PQ} \cong \overline{PR}$	(corresponding sides of congruent triangles)

Theorem 12.1.5 (Converse of Theorem 12.1.4)

Any point inside an angle, equidistant from its arms, is on the bisector of it.

Given

Any point P lies inside $\angle AOB$ such that $\overline{PQ} \cong$ PR, where $\overline{PQ} \perp \overrightarrow{OB}$ and $\overline{PR} \perp \overrightarrow{OA}$.

To Prove

Point P is on the bisector of $\angle AOB$.

Construction

Join P to O.

Proof

Statements	Reasons
In $\Delta POQ \longleftrightarrow \Delta POR$	
$\angle PQO \cong \angle PRO$	given (right angles)
$\overline{PO} \cong \overline{PO}$	common
$\overline{PQ} \cong \overline{PR}$	given
$\therefore$ $\Delta POQ \cong \Delta POR$	H.S. $\cong$ H.S.
Hence $\angle POQ \cong \angle POR$	(corresponding angels of congruent triangles)
i.e., P is on the bisector of $\angle AOB$.	

EXERCSISE 12.2

1. In a quadrilateral ABCD, $\overline{AB} \cong \overline{BC}$ and the right bisectors of $\overline{AD}$, $\overline{CD}$ meet each other at point N. Prove that $\overline{BN}$ is a bisector of $\angle ABC$.
2. The bisectors of $\angle A$, $\angle B$ and $\angle C$ of a quadrilateral ABCP meet each other at point O. Prove that the bisector of $\angle P$ will also pass through the point O.
3. Prove that the right bisectors of congruent sides of an isoscles triangle and its altitude are concurrent.
4. Prove that the altitudes of a triangle are concurrent.

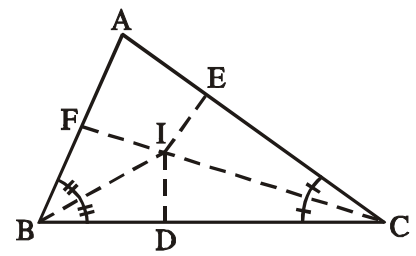
Theorem 12.1.6

The bisectors of the angles of a triangle are concurrent.

Given

ΔABC

To Prove

The bisectors of $\angle A$, $\angle B$ and $\angle C$ are concurrent.

Construction

Draw the bisectors of $\angle B$ and $\angle C$ which intersect at point I. From I, draw $\overline{IF} \perp \overline{AB}$, $\overline{ID} \perp \overline{BC}$ and $\overline{IE} \perp \overline{CA}$.

Proof

Statements	Reasons
$\overline{ID} \cong \overline{IF}$	(Any point on bisector of an angle is equidistant from its arms)
Similarly,	
$\overline{ID} \cong \overline{IE}$	
$\therefore$ $\overline{IE} \cong \overline{IF}$	Each ID, proved.
So, the point I is on the bisector of $\angle A$ ...... (i)	

Also the point *I* is on the bisectors of ∠ABC and ∠BCA ...... (ii) Thus the bisectors of ∠A, ∠B and ∠C are concurrent at I.	Construction {from (i) and (ii)}

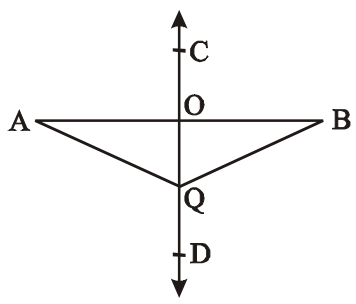
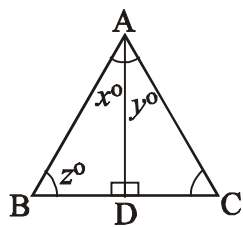
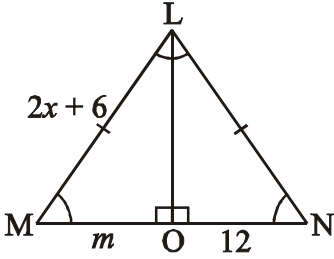
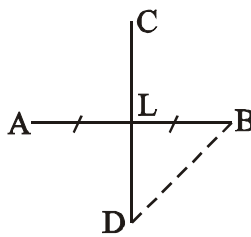
Note. In practical geometry also, by constructing angle bisectors of a triangle, we shall verify that they are concurrent.

EXERCISE 12.3

1. Prove that the bisectors of the angles of base of an isosceles triangle intersect each other on its altitude.
2. Prove that the bisectors of two exterior and third interior angle of a triangle are concurrent.

REVIEW EXERCISE 12

1. Which of the following are true and which are false?
   (i) Bisection means to divide into two equal parts. ......
   (ii) Right bisection of line segment means to draw perpendicular which passes through the mid point. ......
   (iii) Any point on the right bisector of a line segment is not equidistant from its end points. ......
   (iv) Any point equidistant from the end points of a line segment is on the right bisector of it. ......
   (v) The right bisectors of the sides of a triangle are not concurrent. ......
   (vi) The bisectors of the angles of a triangle are concurrent. ......
   (vii) Any point on the bisector of an angle is not equidistant from its arms. ......
   (viii) Any point inside an angle, equidistant from its arms, is on the bisector of it. ......

2. If $\overleftrightarrow{CD}$ is a right bisector of line segment $\overline{AB}$, then
   (i) $m\overline{OA}$ = ...
   (ii) $m\overline{AQ}$ = ...
3. Define the following
   (i) Bisector of a line segment
   (ii) Bisector of an angle
4. The given triangle ABC is equilateral triangle and $\overline{AD}$ is bisector of angle A, then find the values of unknowns x^o, y^o and z^o.

5. In the given congruent triangles LMO and LNO, find the unknowns x and m.

6. $\overline{CD}$ is right bisector of the line segment AB.
   (i) If $m\overline{AB}$ = 6cm, then find the $m\overline{AL}$ and $m\overline{LB}$.
   (ii) If $m\overline{BD}$ = 4cm, then find $m\overline{AD}$.

SUMMARY

In this unit we stated and proved the following theorems:

- Any point on the right bisector of a line segment is equidistant from its end points.
- Any point equidistant from the end points of a line segment is on the right bisector of it.
- The right bisectors of the sides of a triangle are concurrent.
- Any point on the bisector of an angle is equidistant from its arms.
- Any point inside an angle, equidistant from its arms, is on the bisector of it.

- The bisectors of the angles of a triangle are concurrent.
- Right bisection of a line segment means to draw a perpendicular at the mid point of line segment.
- Bisection of an angle means to draw a ray to divide the given angle into two equal parts.

CHAPTER

13

SIDES AND ANGLES OF A TRIANGLE

Animation 13.1: Sides and Angles of a Triangle
Source & Credit: eLearn.punjab

Students Learning Outcomes

After studying this unit, the students will be able to:

- prove that if two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.
- prove that if two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.
- prove that the sum of the lengths of any two sides of a triangle is greater than the length of the third side.
- prove that from a point, out-side a line, the perpendicular is the shortest distance from the point on the line.

Introduction

Recall that if two sides of a triangle are equal, then the angles apposite to them are also equal and vice-versa. But in this unit we shall study some interesting inequality relations among sides and angles of a triangle.

Theorem 13.1.1

If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.

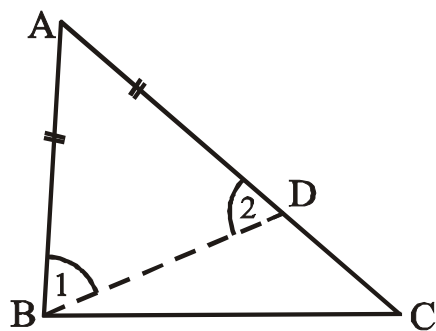
Given

In ΔABC, $m\overline{AC} > m\overline{AB}$

To Prove

$m\angle ABC > m\angle ACB$

Construction

On $\overline{AC}$ take a point D such that $\overline{AD} \cong \overline{AB}$. Join B to D so that ΔADB is an isosceles triangle. Label $\angle 1$ and $\angle 2$ as shown in the given figure.

Proof

Statements	Reasons
In ΔABD	
$m\angle 1 = m\angle 2$ ...... (i)	Angles opposite to congruent sides, (construction)
In ΔBCD, $m\angle ACB < m\angle 2$	
i.e. $m\angle 2 > m\angle ACB$ ...... (ii)	(An exterior angle of a triangle is greater than a non-adjacent interior angle)
$\therefore$ $m\angle 1 > m\angle ACB$ ...... (iii)	By (i) and (ii)
But	
$m\angle ABC = m\angle 1 + m\angle DBC$	Postulate of addition of angles.
$\therefore$ $m\angle ABC > m\angle 1$ ...... (iv)	
$\therefore$ $m\angle ABC > m\angle 1 > m\angle ACB$	By (iii) and (iv)
Hence $m\angle ABC > m\angle ACB$	(Transitive property of inequality of real numbers)

Example 1

Prove that in a scalene triangle, the angle opposite to the largest side is of measure greater than 60°. (i.e., two-third of a right-angle)

Given

In ABC, $m\overline{AC} > m\overline{AB}$ $m\overline{AC}$, $m\overline{AB} > m\overline{BC}$.

To Prove

$m\angle B > 60°$.

Proof

Statements	Reasons
In ΔABC	
$m\angle B > m\angle C$	$m\overline{AC} > m\overline{AB}$ (given)
$m\angle B > m\angle A$	$m\overline{AC} > m\overline{BC}$ (given)
But $m\angle A + m\angle B + m\angle C = 180°$	$\angle A, \angle B, \angle C$ are the angles of ΔABC
$\therefore$ $m\angle B + m\angle B + m\angle B > 180°$	$m\angle B > m\angle C$, $m\angle B > m\angle A$ (proved)
Hence $m\angle B > 60°$	$180°/3 = 60°$

Example 2

In a quadrilateral ABCD, $\overline{AB}$ is the longest side and $\overline{CD}$ is the shortest side. Prove that $m\angle BCD > m\angle BAD$.

Given

In quad. ABCD, $\overline{AB}$ is the longest side and $\overline{CD}$ is the shortest side.

To Prove

$m\angle BCD > m\angle BAD$

Construction

Joint A to C.

Name the angles $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$ as shown in the figure.

Proof

Statements	Reasons
In ΔABC, $m\angle 4 > \angle 2$ I	$m\overline{AB} > m\overline{BC}$ (given)
In ΔACD, $m\angle 3 > m\angle 1$ II	$m\overline{AD} > m\overline{CD}$ (given)
$\therefore$ $m\angle 4 + m\angle 3 > m\angle 2 + m\angle 1$	From I and II
Hence $m\angle BCD > m\angle BAD$	$\because$ $\begin{cases} m\angle 4 + m\angle 3 = m\angle BCD \\ m\angle 2 + m\angle 1 = m\angle BAD \end{cases}$

Theorem 13.1.2

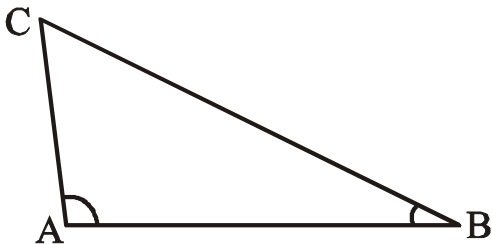
(Converse of Theorem 13.1.1)

If two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.

Given

In ΔABC, $m\angle A > m\angle B$

To Prove

$m\overline{BC} > m\overline{AC}$

Proof

Statements	Reasons
If, $m\overline{BC} \not> m\overline{AC}$, then either (i) $m\overline{BC} = m\overline{AC}$ or (ii) $m\overline{BC} < m\overline{AC}$	(Trichotomy property of real numbers)
From (i) if $m\overline{BC} = m\overline{AC}$, then $m\angle A = m\angle B$	(Angles opposite to congruent sides are congruent)
which is not possible.	Contrary to the given.
From (ii) if $mBC < mAC$, then $m\angle A < m\angle B$	(The angle opposite to longer side is greater than angle opposite to smaller side)
This is also not possible.	Contrary to the given.
$\therefore$ $m\overline{BC} \neq m\overline{AC}$ and $m\overline{BC} \not< m\overline{AC}$	
Thus $m\overline{BC} > m\overline{AC}$	Trichotomy property of real numbers.

Corollaries

(i) The hypotenuse of a right angled triangle is longer than each of the other two sides.

(ii) In an obtuse angled triangle, the side opposite to the obtuse angle is longer than each of the other two sides.

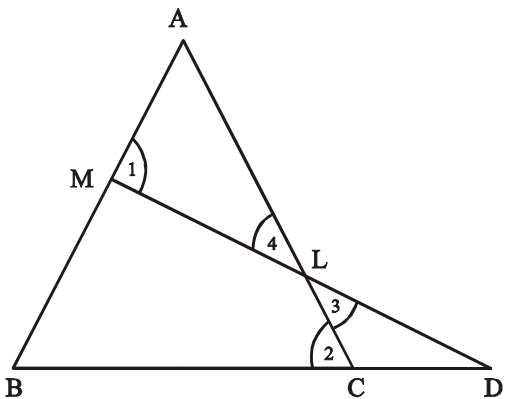
Example

ABC is an isosceles triangle with base $\overline{BC}$. On $\overrightarrow{BC}$ a point D is taken away from C. A line segment through D cuts $\overline{AC}$ at L and $\overline{AB}$ at M. Prove that $m\overline{AL} > m\overline{AM}$.

Given

In ΔABC, $\overline{AB} \cong \overline{AC}$

D is a point on $\overrightarrow{BC}$ away from C.

A line segment through D cuts $\overline{AC}$ at L and $\overline{AB}$ at M.

To Prove

$m\overline{AL} > m\overline{AM}$

Proof

Statements	Reasons
In ΔABC	
$\angle B \cong \angle 2$ I	$\overline{AB} \cong \overline{AC}$ (given)
In ΔMBD	
$m\angle 1 > m\angle B$ II	($\angle 1$ is an ext. $\angle$ and $\angle B$ is its internal opposite $\angle$)
$\therefore$ $m\angle 1 > m\angle 2$ III	From I and II
In ΔLCD,	
$m\angle 2 > m\angle 3$ IV	($\angle 2$ is an ext. $\angle$ and $\angle 3$ is its internal opposite $\angle$)
$\therefore$ $m\angle 1 > m\angle 3$ V	From III and IV
But $\angle 3 \cong \angle 4$ VI	Vertical angles
$\therefore$ $m\angle 1 > m\angle 4$	From V and VI
Hence $m\overline{AL} > m\overline{AM}$	In ΔALM, $m\angle 1 > m\angle 4$ (proved)

Theorem 13.1.3

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Given

ΔABC

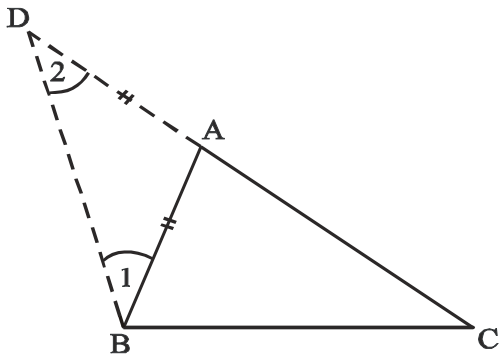
To Prove

(i) $m\overline{AB} + m\overline{AC} > m\overline{BC}$

(ii) $m\overline{AB} + m\overline{BC} > m\overline{AC}$

(iii) $m\overline{BC} + m\overline{CA} > m\overline{AB}$

Construction

Take a point D on $\overrightarrow{CA}$ such that $\overline{AD} \cong \overline{AB}$. Join B to D and name the angles. $\angle 1$, $\angle 2$ as shown in the given figure.

Proof

Statements	Reasons
In ΔABD,	
$\angle 1 \cong \angle 2$ ....(i)	$\overline{AD} \cong \overline{AB}$ (construction)
$m\angle DBC > m\angle 1$ ....(ii)	$m\angle DBC = m\angle 1 + m\angle ABC$
$\therefore$ $m\angle DBC > m\angle 2$ ....(iii)	From (i) and (ii)
In ΔDBC	
$m\overline{CD} > m\overline{BC}$	By (iii)
i.e., $m\overline{AD} + m\overline{AC} > m\overline{BC}$	$m\overline{CD} = m\overline{AD} + m\overline{AC}$
Hence $m\overline{AB} + m\overline{AC} > m\overline{BC}$	$m\overline{AD} = m\overline{AB}$ (construction)
Similarly,	
$m\overline{AB} + m\overline{BC} > m\overline{AC}$	
and	
$m\overline{BC} + m\overline{CA} > m\overline{AB}$	

Example 1

Which of the following sets of lengths can be the lengths of the sides of a triangle?

(a) 2 cm, 3 cm, 5 cm (b) 3 cm, 4 cm, 5 cm, (c) 2 cm, 4 cm, 7 cm,

(a) $\because$ $2 + 3 = 5$

$\therefore$ This set of lengths cannot be those of the sides of a triangle.

(b) $\because$ $3 + 4 > 5$, $3 + 5 > 4$, $4 + 5 > 3$

$\therefore$ This set can form a triangle

(c) $\because$ $2 + 4 < 7$

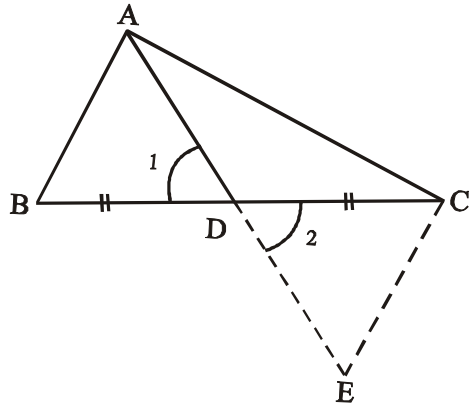
$\therefore$ This set of lengths cannot be the sides of a triangle.

Example 2

Prove that the sum of the measures of two sides of a triangle is greater than twice the measure of the median which bisects the third side.

Given

In ΔABC,

median $\overline{AD}$ bisects side $\overline{BC}$ at D.

To Prove

$m\overline{AB} + m\overline{AC} > 2m\overline{AD}$.

Construction

On $\overrightarrow{AD}$ take a point E, such that $\overline{DE} \cong \overline{AD}$. Join C to E. Name the angles $\angle 1$, $\angle 2$ as shown in the figure.

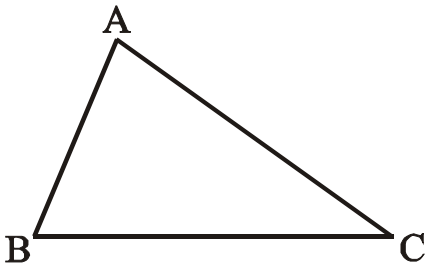
Proof

Statements	Reasons
In $\Delta ABD \longleftrightarrow \Delta ECD$	
$\overline{BD} \cong \overline{CD}$	Given
$\angle 1 \cong \angle 2$	Vertical angles
$\overline{AD} \cong \overline{ED}$	Construction
$\Delta ABD \cong \Delta ECD$	S.A.S. Postulate
$\overline{AB} \cong \overline{EC}$ I	Corresponding sides of $\cong \Delta$s
$m\overline{AC} + m\overline{EC} > m\overline{AE}$ II	ACE is a triangle
$m\overline{AC} + m\overline{AB} > m\overline{AE}$	From I and II
Hence $m\overline{AC} + m\overline{AB} > 2m\overline{AD}$	$m\overline{AE} = 2m\overline{AD}$ (construction)

Example 3

Prove that the difference of measures of two sides of a triangle is less than the measure of the third side.

Given

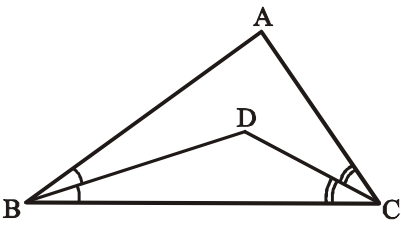
ΔABC

To Prove

$m\overline{AC} - m\overline{AB} < m\overline{BC}$
$m\overline{BC} - m\overline{AB} < m\overline{AC}$
$m\overline{BC} - m\overline{AC} > m\overline{AB}$

Proof:

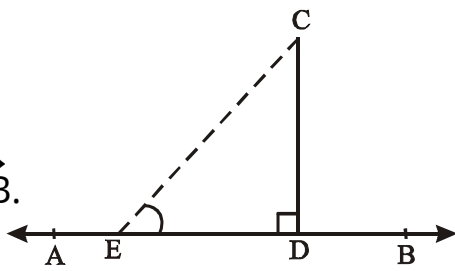
Statements	Reasons
$m\overline{AB} + m\overline{BC} > m\overline{AC}$	ABC is a triangle
$(m\overline{AB} + m\overline{BC} - m\overline{AB})$ $>(m\overline{AC} - m\overline{AB})$	Subtracting $m\overline{AB}$ from both sides
$\therefore$ $m\overline{BC} > (m\overline{AC} - m\overline{AB})$	
or $m\overline{AC} - m\overline{AB} < m\overline{BC}$ I	$a > b \Rightarrow b < a$
Similarly	
$m\overline{BC} - m\overline{AB} < m\overline{AC}$ $m\overline{BC} - m\overline{AC} < m\overline{AB}$	Reason similar to I

EXERCISE 13.1

1. Two sides of a triangle measure 10 cm and 15 cm. Which of the following measure is possible for the third side?
   (a) 5 cm (b) 20 cm (c) 25 cm (d) 30 cm
2. O is an interior point of the ΔABC. Show that
   $m\overline{OA} + m\overline{OB} + m\overline{OC} > \frac{1}{2}(m\overline{AB} + m\overline{BC} + m\overline{CA})$
3. In the Δ ABC, $m\angle B = 70°$ and $m\angle C = 45°$. Which of the sides of the triangle is longest and which is the shortest?
4. Prove that in a right-angled triangle, the hypotenuse is longer than each of the other two sides.
5. In the triangular figure, $m\overline{AB} > m\overline{AC}$. $\overline{BD}$ and $\overline{CD}$ are the bisectors of B and C respectively. Prove that $m\overline{BD} > m\overline{DC}$.

Theorem 13.1.4

From a point, outside a line, the perpendicular is the shortest distance from the point to the line.

Given

A line AB and a point C (not lying on $\overleftrightarrow{AB}$) and a point D on $\overleftrightarrow{AB}$ such that $\overline{CD} \perp \overleftrightarrow{AB}$.

To Prove

$m\overline{CD}$ is the shortest distance form the point C to $\overleftrightarrow{AB}$.

Construction

Take a point E on $\overleftrightarrow{AB}$. Join C and E to form a ΔCDE.

Proof

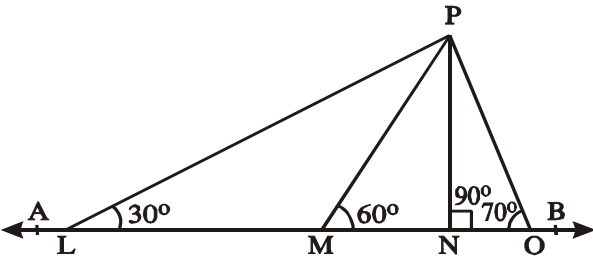
Statements	Reasons
In ΔCDE	
$m\angle CDB > m\angle CED$	(An exterior angle of a triangle is greater than non adjacent interior angle).
But $m\angle CDB = m\angle CDE$	Supplement of right angle.
$\therefore$ $m\angle CDE > m\angle CED$	
or $m\angle CED < m\angle CDE$	$a > b \Rightarrow b < a$
or $m\overline{CD} < m\overline{CE}$	Side opposite to greater angle is greater.
But E is any point on $\overleftrightarrow{AB}$	
Hence $m\overline{CD}$ is the shortest distance from C to $\overleftrightarrow{AB}$.	

Note:

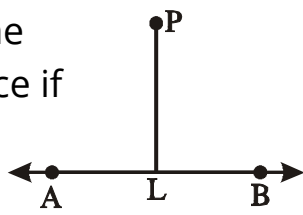
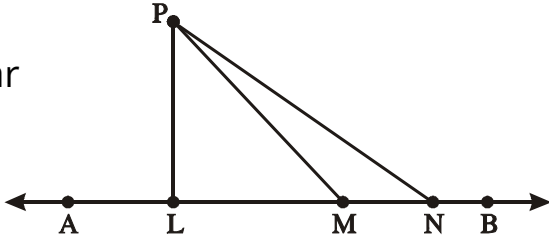
(i) The distance between a line and a point not on it, is the length of the perpendicular line segment from the point to the line.

(ii) The distance between a line and a point lying on it is zero.

EXERCISE 13.2

1. In the figure, P is any point and AB is a line. Which of the following is the shortest distance between the point P and the line AB?

(a) $m\overline{PL}$ (b) $m\overline{PM}$ (c) $m\overline{NP}$ (d) $m\overline{PO}$

2. In the figure, P is any point lying away from the line AB. Then $m\overline{PL}$ will be the shortest distance if

(a) $m\angle PLA = 80°$ (b) $m\angle PLB = 100°$
(c) $m\angle PLA = 90°$

3. In the figure, $\overline{PL}$ is prependicular to the line $\overline{AB}$ and $m\overline{LN} > m\overline{LM}$. Prove that $m\overline{PN} > m\overline{PM}$.

REVIEW EXERCISE 13

1. Which of the following are true and which are false?
   (i) The angle opposite to the longer side is greater. ......
   (ii) In a right-angled triangle greater angle is of 60°. ......
   (iii) In an isosceles right-angled triangle, angles other than right angle are each of 45°. ......
   (iv) A triangle having two congruent sides is called equilateral triangle. ......
   (v) A perpendicular from a point to line is shortest distance. ...
   (vi) Perpendicular to line form an angle of 90°. ......
   (vii) A point out side the line is collinear. ......
   (viii) Sum of two sides of triangle is greater than the third. ......
   (ix) The distance between a line and a point on it is zero. ......
   (x) Triangle can be formed of lengths 2 cm, 3 cm and 5 cm. ...
2. What will be angle for shortest distance from an outside point to the line?
3. If 13 cm, 12 cm, and 5 cm are the lengths of a triangle, then verify that difference of measures of any two sides of a triangle is less than the measure of the third side.
4. If 10 cm, 6 cm and 8 cm are the lengths of a triangle, then verify that sum of measures of two sides of a triangle is greater than the third side.
5. 3 cm, 4 cm and 7 cm are not the lengths of the triangle. Give the reason.

6. If 3 cm and 4 cm are lengths of two sides of a right angle triangle, then what should be the third length of the triangle.

SUMMARY

In this unit we stated and proved the following theorems:

- If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.
- If two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.
- The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
- From a point, outside a line, the perpendicular is the shortest distance from the point to the line.

CHAPTER

14

RATIO AND PROPORTION

Animation 14.1: Ratio and Proportion
Source & Credit: eLearn.punjab

Students Learning Outcomes

After studying this unit, the students will be able to:

- prove that a line parallel to one side of a triangle, intersecting the other two sides, divides them proportionally.
- prove that if a line segment intersects the two sides of a triangle in the same ratio, then it is parallel to the third side.
- prove that the internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the lengths of the sides containing the angle.
- prove that if two triangles are similar, the measures of their corresponding sides are proportional

Introduction

In this unit we will prove some theorems and corollaries involving ratio and proportions of sides of triangle and similarity of triangles. A knowledge of ratio and proportion is necessary requirement of many occupations like food service occupation, medications in health, preparing maps for land survey and construction works, profit to cost ratios etc.

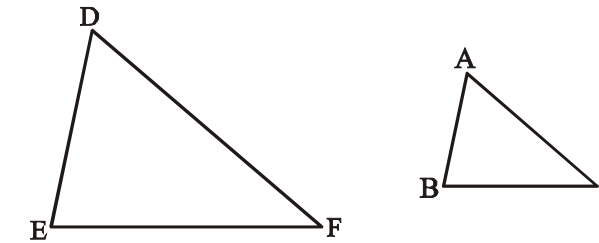
Recall that we defined ratio $a : b = \frac{a}{b}$ as the comparison of two alike quantities a and b, called the elements (terms) of a ratio. (Elements must be expressed in the same units). Equality of two ratios was defined as proportion.

That is, if $\boldsymbol{a : b = c : d}$, then $\boldsymbol{a, b, c}$ and $\boldsymbol{d}$ are said to be in proportion.

Similar Triangles

Equally important are the similar shapes. In particular the similar triangles that have many practical applications. For example, we know that a photographer can develop prints of different sizes from the same negative. In spite of the difference in sizes, these pictures look like each other. One photograph is simply an enlargement of another. They are said to be similar in shape. Geometrical figures can also be similar. e.g., If

In $\Delta ABC \longleftrightarrow \Delta DEF$

$\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$, and $\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{BC}}{m\overline{EF}} = \frac{m\overline{CA}}{m\overline{FD}}$

then ΔABC and ΔDEF are called similar triangles which is symbolically written as

$\Delta ABC \sim \Delta DEF$

It means that corresponding angles of similar triangles are equal and measures of their corresponding sides are proportional.

$\Delta PQR \cong \Delta LMN$ means that in

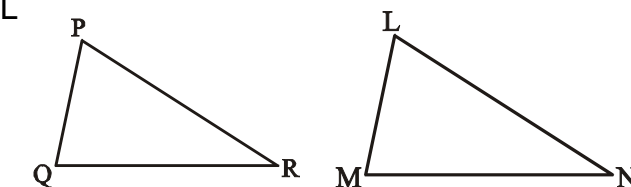
$\Delta PQR \longleftrightarrow \Delta LMN$

$\angle P \cong \angle L$, $\angle Q \cong \angle M$,

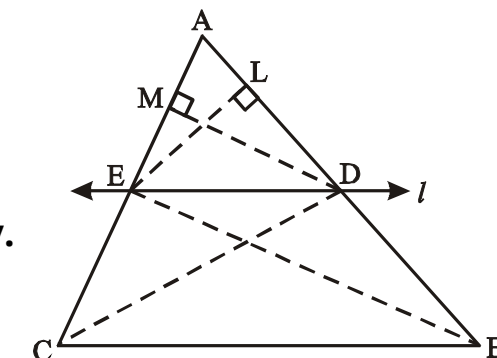
$\angle R \cong \angle N$, $\overline{PQ} \cong \overline{LM}$,

$\overline{QR} \cong \overline{MN}$, $\overline{RP} \cong \overline{NL}$

Now as $\frac{m\overline{PQ}}{m\overline{LM}} = \frac{m\overline{QR}}{m\overline{MN}} = \frac{m\overline{RP}}{m\overline{NL}} = 1$

$\therefore \quad \Delta PQR \sim \Delta LMN$

In other words, two congruent triangles are similar also. But two similar triangles are not necessarily congruent, as congruence of their corresponding sides is not necessary.

Theorem 14.1.1

A line parallel to one side of a triangle and intersecting the other two sides divides them proportionally.

Given

In ΔABC, the line ℓ is intersecting the sides $\overline{AC}$ and $\overline{AB}$ at points E and D respectively such that $\overline{ED} \parallel \overline{CB}$.

To Prove

$m\overline{AD} : m\overline{DB} = m\overline{AE} : m\overline{EC}$

Construction

Join B to E and C to D. From D draw $\overline{DM} \perp \overline{AC}$ and from E draw $\overline{EL} \perp \overline{AB}$.

Proof

Statements	Reasons
In triangles BED and AED, $\overline{EL}$ is the common perpendicular.	
$\therefore$ Area of $\Delta BED = \frac{1}{2} \times m\overline{BD} \times m\overline{EL}$ .... (i)	Area of a $\Delta = \frac{1}{2}$(base)(height)
and Area of $\Delta AED = \frac{1}{2} \times m\overline{AD} \times m\overline{EL}$ .... (ii)	
Thus $\frac{\text{Area of } \Delta BED}{\text{Area of } \Delta AED} = \frac{m\overline{BD}}{m\overline{AD}}$ .... (iii)	Dividing (i) by (ii)
Similarly	
$\frac{\text{Area of } \Delta CDE}{\text{Area of } \Delta ADE} = \frac{m\overline{EC}}{m\overline{AE}}$ .... (iv)	
But $\Delta BED \cong \Delta CDE$	(Areas of triangles with common base and same altitudes are equal). Given that $\overline{ED} \parallel m\overline{CB}$, so altitudes are equal.
$\therefore$ From (iii) and (iv), we have	
$\frac{m\overline{DB}}{m\overline{AD}} = \frac{m\overline{EC}}{m\overline{AE}}$ or $\frac{m\overline{AD}}{m\overline{DB}} = \frac{m\overline{AE}}{m\overline{EC}}$	Taking reciprocal of both sides.
Hence $m\overline{AD} : m\overline{DB} = m\overline{AE} : m\overline{EC}$	

Observe that

From the above theorem we also have

$$\frac{m\overline{BD}}{m\overline{AB}} = \frac{m\overline{CE}}{m\overline{AC}} \text{ and } \frac{m\overline{AD}}{m\overline{AB}} = \frac{m\overline{AE}}{m\overline{AC}}$$

Corollaries

(a) If $\frac{m\overline{AD}}{m\overline{AB}} = \frac{m\overline{AE}}{m\overline{AC}}$, then $\overline{DE} \parallel \overline{BC}$ (b) If $\frac{m\overline{AB}}{m\overline{DB}} = \frac{m\overline{AC}}{m\overline{EC}}$, then $\overline{DE} \parallel \overline{BC}$

Points to be noted

(i) Two points determine a line and three non-collinear points determine a plane.
(ii) A line segment has exactly one midpoint.
(iii) If two intersecting lines form equal adjacent angles, the lines are perpendicular.

Theorem 14.1.2

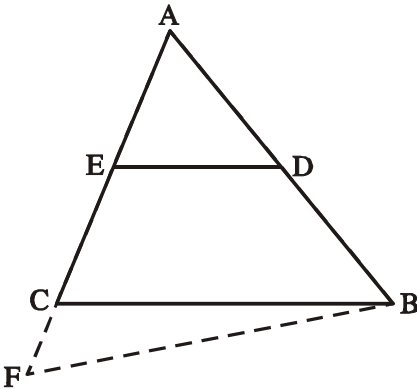
(Converse of Theorem 14.1.1)

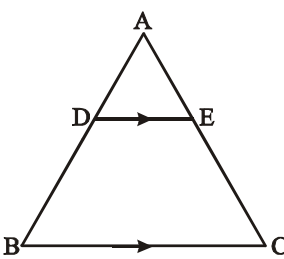
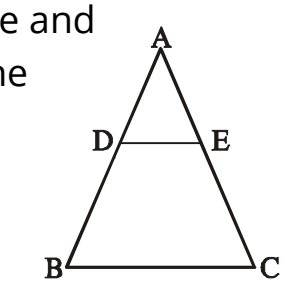
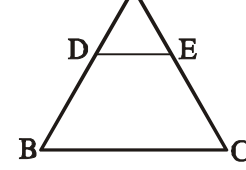
If a line segment intersects the two sides of a triangle in the same ratio, then it is parallel to the third side.

Given

In ΔABC, $\overline{ED}$ intersects $\overline{AB}$ and $\overline{AC}$ such that $m\overline{AD} : m\overline{DB} = m\overline{AE} : m\overline{EC}$

To Prove

$\overline{ED} \parallel \overline{CB}$

Construction

If $\overline{ED} \nparallel \overline{CB}$, then draw $\overline{BF} \parallel \overline{DE}$ to meet $\overline{AC}$ produced at F.

Proof

Statements	Reasons
In ΔABF	

$\overline{DE} \parallel \overline{BF}$	Construction
$\therefore \quad \dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{m\overline{AE}}{m\overline{EF}}$ ....(i)	(A line parallel to one side of a triangle divides the other two sides proportionally Theorem 14.1.1)
But $\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{m\overline{AE}}{m\overline{EC}}$ ....(ii)	Given
$\therefore \quad \dfrac{m\overline{AE}}{m\overline{EF}} = \dfrac{m\overline{AE}}{m\overline{EC}}$	From (i) and (ii)
or $\quad m\overline{EF} = m\overline{EC}$, which is possible only if point F is coincident with C.	(Property of real numbers.)
$\therefore$ Our supposition is wrong Hence $\overline{ED} \parallel \overline{CB}$	

EXERCISE 14.1

1. In ΔABC, $\overline{DE} \parallel \overline{BC}$.
   (i) If $m\overline{AD} = 1.5\text{cm}$, $m\overline{BD} = 3\text{cm}$, $m\overline{AE} = 1.3\text{cm}$, then find $m\overline{CE}$.
   (ii) If $m\overline{AD} = 2.4\text{cm}$, $m\overline{AE} = 3.2\text{cm}$, $m\overline{EC} = 4.8\text{cm}$, find $m\overline{AB}$.

   (iii) If $\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{3}{5}$ and $m\overline{AC} = 4.8\text{cm}$, find $m\overline{AE}$.
   (iv) If $m\overline{AD} = 2.4\text{cm}$, $m\overline{AE} = 3.2\text{cm}$, $m\overline{DE} = 2\text{cm}$, $m\overline{BC} = 5\text{cm}$, find $m\overline{AB}$, $m\overline{DB}$, $m\overline{AC}$, $m\overline{CE}$.
   (v) If $\overline{AD} = 4x - 3$, $\overline{AE} = 8x - 7$, $\overline{BD} = 3x - 1$, and $\overline{CE} = 5x - 3$, find the value of x.
2. If ΔABC is an isosceles triangle, $\angle A$ is vertex angle and $\overline{DE}$ intersects the sides $\overline{AB}$ and $\overline{AC}$ as shown in the figure so that
   $m\overline{AD} : m\overline{DB} = m\overline{AE} : m\overline{EC}$.

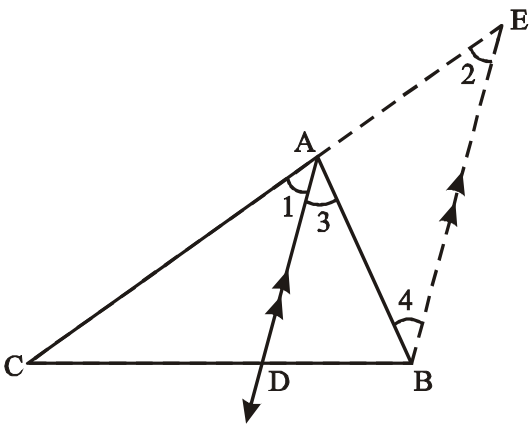
   Prove that ΔADE is also an isosceles triangle.

3. In an equilateral triangle ABC shown in the figure, $m\overline{AE} : m\overline{AC} = m\overline{AD} : m\overline{AB}$

   Find all the three angles of ΔADE and name it also.

4. Prove that the line segment drawn through the mid-point of one side of a triangle and parallel to another side bisects the third side.
5. Prove that the line segment joining the mid-points of any two sides of a triangle is parallel to the third side.

Theorem 14.1.3

The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the lengths of the sides containing the angle.

Given

In ΔABC internal angle bisector of $\angle A$ meets $\overline{CB}$ at the point D.

To Prove

$m\overline{BD} : m\overline{DC} = m\overline{AB} : m\overline{AC}$

Construction

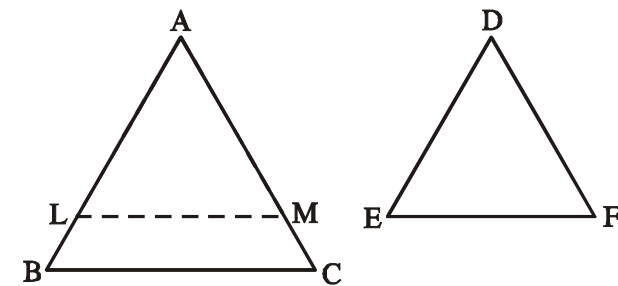
Draw a line segment $\overline{BE} \parallel \overline{DA}$ to meet $\overline{CA}$ produced at E.

Proof

Statements	Reasons
$\because$ $\overline{AD} \parallel \overline{EB}$ and $\overline{EC}$ intersects them,	Construction
$\therefore \quad m\angle 1 = m\angle 2$ ......(i)	Corresponding angles
Again $\overline{AD} \parallel \overline{EB}$	

and $\overline{AB}$ intersects them.	
$\therefore \quad m\angle 3 = m\angle 4$ ....(ii)	Alternate angles
But $m\angle 1 = m\angle 3$	Given
$\therefore \quad m\angle 2 = m\angle 4$	From (i) and (ii)
and $\overline{AB} \cong AE$ or $\overline{AE} \cong \overline{AB}$	In a Δ, the sides opposite to congruent angles are also congruent.
Now $\overline{AD} \parallel \overline{EB}$	Construction
$\therefore \quad \frac{m\overline{BD}}{m\overline{DC}} = \frac{m\overline{EA}}{m\overline{AC}}$	by Theorem 14.1.1
or $\quad \frac{m\overline{BD}}{m\overline{DC}} = \frac{m\overline{AE}}{m\overline{AC}}$	$m\overline{EA} = m\overline{AB}$ (proved)
Thus $m\overline{BD} : m\overline{DC} = m\overline{AB} : \overline{AC}$	

Theorem 14.1.4

If two triangles are similar, then the measures of their corresponding sides are proportional.

Given

$\Delta ABC \sim \Delta DEF$

i.e., $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $\angle C \cong \angle F$

To Prove

$$\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{AC}}{m\overline{DF}} = \frac{m\overline{BC}}{m\overline{EF}}$$

14. Ratio and Proportion eLearn.Punjab

Construction

(i) Suppose that $m\overline{AB} > m\overline{DE}$

(ii) $m\overline{AB} \leq m\overline{DE}$

On $\overline{AB}$ take a point L such that $m\overline{AL} = m\overline{DE}$.

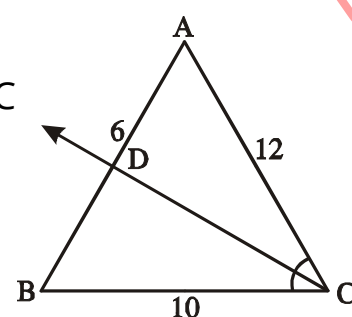
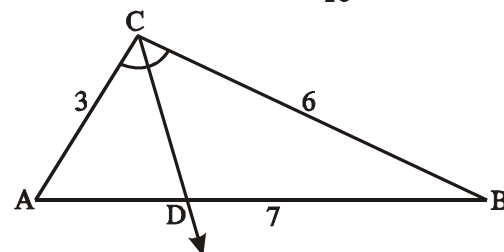
On $\overline{AC}$ take a point M such that $m\overline{AM} = m\overline{DF}$. Join L and M by the line segment $\overline{LM}$.

Proof

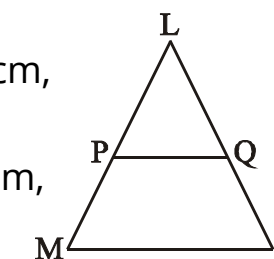
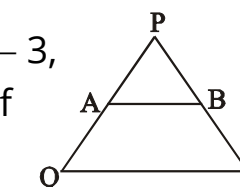
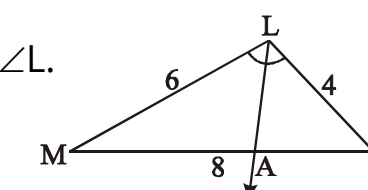
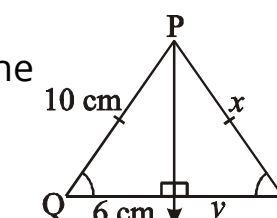
Statements	Reasons
(i) In $\Delta ALM \leftrightarrow \Delta DEF$	
$\angle A \cong \angle D$	Given
$\overline{AL} \cong \overline{DE}$	Construction
$\overline{AM} \cong \overline{DF}$	Construction
Thus $\Delta ALM \cong \Delta DEF$	S.A.S. Postulate
and $\angle L \cong \angle E$, $\angle M \cong \angle F$	(Corresponding angles of congruent triangles)
Now $\angle E \cong \angle B$ and $\angle F \cong \angle C$	Given
$\therefore \quad \angle L \cong \angle B$, $\angle M \cong \angle C$	Transitivity of congruence
Thus $\overline{LM} \parallel \overline{BC}$	Corresponding angles are equal.
Hence $\frac{m\overline{AL}}{m\overline{AB}} = \frac{m\overline{AM}}{m\overline{AC}}$	by Theorem 14.1.1
or $\frac{m\overline{DE}}{m\overline{AB}} = \frac{m\overline{DF}}{m\overline{AC}}$ ....(i)	$m\overline{AL} = m\overline{DE}$ and $m\overline{AM} = m\overline{DF}$ (construction)
Similarly by intercepting segments on $\overline{BA}$ and $\overline{BC}$, we can prove that	
$\frac{m\overline{DE}}{m\overline{AB}} = \frac{m\overline{EF}}{m\overline{BC}}$ .....(ii)	
Thus $\frac{m\overline{DE}}{m\overline{AB}} = \frac{m\overline{DF}}{m\overline{AC}} = \frac{m\overline{EF}}{m\overline{BC}}$	by (i) and (ii)
or $\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{AC}}{m\overline{DF}} = \frac{m\overline{BC}}{m\overline{EF}}$	by taking reciprocals
(ii) If $m\overline{AB} < m\overline{DE}$, it can	

similarly be proved by taking intercepts on the sides of ΔDEF.	
If $m\overline{AB} = m\overline{DE}$,	
then $\Delta ABC \leftrightarrow \Delta DEF$	
$\angle A \cong \angle D$	Given
$\angle B \cong \angle E$	Given
and $\overline{AB} \cong \overline{DE}$	
so $\Delta ABC \cong \Delta DEF$	A.S.A. $\cong$ A.S.A.
Thus $\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{AC}}{m\overline{DF}} = \frac{m\overline{BC}}{m\overline{EF}} = 1$	$\overline{AC} \cong \overline{DF}$, $\overline{BC} \cong \overline{EF}$
Hence the result is true for all cases.	

EXERCISE 14.2

1. In ΔABC as shown in the figure, $\overrightarrow{CD}$ bisects $\angle C$ and meets $\overline{AB}$ at D. $m\overline{BD}$ is equal to
   (a) 5 (b) 16 (c) 10 (d) 18

2. In ΔABC shown in the figure, $\overrightarrow{CD}$ bisects $\angle C$. If $m\overline{AC} = 3$, $m\overline{CB} = 6$ and $m\overline{AB} = 7$, then find $m\overline{AD}$ and $m\overline{DB}$.

3. Show that in any correspondence of two triangles, if two angles of one triangle are congruent to the corresponding angles of the other, then the triangles are similar.
4. If line segments AB and CD are intersecting at point X and $\frac{m\overline{AX}}{m\overline{XB}} = \frac{m\overline{CX}}{m\overline{XD}}$ then show that ΔAXC and ΔBXD are similar.

REVIEW EXERCISE 14

1. Which of the following are true and which are false?
   (i) Congruent triangles are of same size and shape. ......
   (ii) Similar triangles are of same shape but different sizes. .....
   (iii) Symbol used for congruent is '~'. ......
   (iv) Symbol used for similarity is '≅'. ......
   (v) Congruent triangles are similar. ......
   (vi) Similar triangles are congruent. ......
   (vii) A line segment has only one mid point. ......
   (viii) One and only one line can be drawn through two points. ......
   (ix) Proportion is non-equality of two ratios. ......
   (x) Ratio has no unit. ......
2. Define the following:
   (i) Ratio (ii) Proportion
   (iii) Congruent Triangles (iv) Similar Triangles
3. In ΔLMN shown in the figure, $\overline{MN} \parallel \overline{PQ}$
   (i) If $m\overline{LM} = 5$cm, $m\overline{LP} = 2.5$cm, $m\overline{LQ} = 2.3$cm, then find $m\overline{LN}$.
   (ii) If $m\overline{LM} = 6$cm, $m\overline{LQ} = 2.5$cm, $m\overline{QN} = 5$cm, then find $m\overline{LP}$.
4. In the shown figure, let $m\overline{PA} = 8x - 7$, $m\overline{PB} = 4x - 3$, $m\overline{AQ} = 5x - 3$, $m\overline{BR} = 3x - 1$. Find the value of x if $\overline{AB} \parallel \overline{QR}$.

5. In ΔLMN shown in the figure, $\overrightarrow{LA}$ bisects $\angle L$. If $m\overline{LN} = 4$, $m\overline{LM} = 6$, $m\overline{MN} = 8$, then find $m\overline{MA}$ and $m\overline{AN}$.

6. In isosceles ΔPQR shown in the figure, find the value of x and y.

SUMMARY

In this unit we stated and proved the following theorems and gave some necessary definitions:

- A line parallel to one side of a triangle and intersecting the other two sides divides them proportionally.
- If a line segment intersects the two sides of a triangle in the same ratio, then it is parallel to the third side.
- The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the lengths of the sides containing the angle.
- If two triangles are similar, then the measures of their corresponding sides are proportional.
- The ratio between two alike quantities is defined as $a : b = \frac{a}{b}$, where a and b are the elements of the ratio.
- Proportion is defined as the equality of two ratios i.e., $a : b = c : d$.
- Two triangles are said to be similar if they are equiangular and corresponding sides are proportional.

CHAPTER

15 PYTHAGORAS' THEOREM

Animation 15.1: Pythagoras-2a
Source & Credit: wikipedia

Students Learning Outcomes

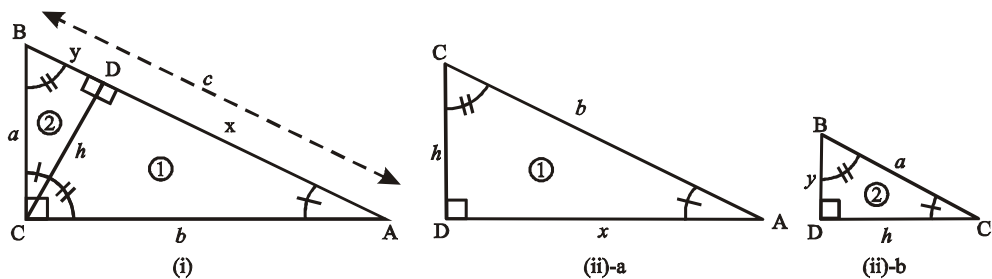
After studying this unit, the students will be able to:

- prove that in a right-angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides. (Pythagoras' theorem).
- prove that if the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right angled triangle (converse to Pythagoras' theorem).

Introduction

Pythagoras, a Greek philosopher and mathematician discovered the simple but important relationship between the sides of a right-angled triangle. He formulated this relationship in the form of a theorem called Pythagoras' Theorem after his name. There are various methods of proving this theorem. We shall prove it by using similar triangles. We shall state and prove its converse also and then apply them to solve different problems.

Pythagoras Theorem 15.1.1

In a right angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

Given

ΔACB is a right angled triangle in which $m\angle C = 90^0$ and $m\overline{BC} = a$, $m\overline{AC} = b$ and $m\overline{AB} = c$.

To Prove

$c^2 = a^2 + b^2$

Construction

Draw $\overline{CD}$ perpendicular from C on $\overline{AB}$.

Let $m\overline{CD} = h$, $m\overline{AD} = x$ and $m\overline{BD} = y$. Line segment CD splits ΔABC into two Δs ADC and BDC which are separately shown in the figures (ii) –a and (ii) –b respectively.

Proof (Using similar Δs)

Statements	Reasons
In $\Delta ADC \leftrightarrow \Delta ACB$	Refer to figure (ii) -a and (i)
$\angle A \cong \angle A$	
$\angle ADC \cong \angle ACB$	Construction – given, each angle = 90^0
$\angle C \cong \angle B$	$\angle C$ and $\angle B$, complements of $\angle A$
$\therefore$ $\Delta ADC \sim \Delta ACB$	Congruency of three angles
$\therefore$ $\frac{x}{b} = \frac{b}{c}$	(Measures of corresponding sides of similar triangles are proportional)
or $x = \frac{b^2}{c}$ .....(I)	
Again in $\Delta BDC \leftrightarrow \Delta BCA$	Refer to figure (ii)-b and (i)
$\angle B \cong \angle B$	Common - self congruent
$\angle BDC \cong \angle BCA$	Construction – given, each angle = 90^0
$\angle C \cong \angle A$	$\angle C$ and $\angle A$, complements of $\angle B$
$\therefore$ $\Delta BDC \sim \Delta BCA$	Congruency of three angles
$\therefore$ $\frac{y}{a} = \frac{a}{c}$	(Corresponding sides of similar triangles are proportional)
$\therefore$ $y = \frac{a^2}{c}$ .....(II)	
But $y + x = c$	Supposition.
$\therefore$ $\frac{a^2}{c} + \frac{b^2}{c} = c$	By (I) and (II)
or $a^2 + b^2 = c^2$	Multiplying both sides by c.
i.e., $c^2 = a^2 + b^2$	

Corollary

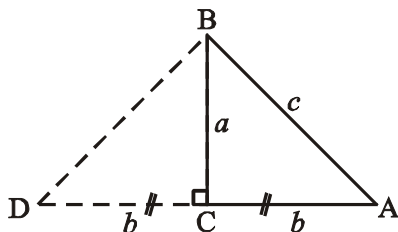
In a right angled ΔABC, right angle at A,

(i) $\overline{AB}^2 = \overline{BC}^2 - \overline{CA}^2$
(ii) $\overline{AC}^2 = \overline{BC}^2 - \overline{AB}^2$

Remark

Pythagoras' Theorem has many proofs. The one we have given is based on the proportionality of the sides of two similar triangles. For convenience Δs ADC and CDB have been shown separately. Otherwise, the theorem is usually proved using figure (i) only.

Theorem 15.1.2 [Converse of Pythagoras' Theorem 15.1.1]

If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right angled triangle.

Given

In a ΔABC, m$\overline{AB}$ = c, m$\overline{BC}$ = a and m$\overline{AC}$ = b such that $a^2 + b^2 = c^2$.

To Prove

ΔACB is a right angled triangle.

Construction

Draw $\overline{CD}$ perpendicular to $\overline{BC}$ such that $\overline{CD} \cong \overline{CA}$. Join the points B and D.

Proof

Statements	Reasons
ΔDCB is a right-angled triangle.	Construction
$\therefore$ $(m\overline{BD})^2 = a^2 + b^2$	Pythagoras theorem
But $a^2 + b^2 = c^2$	Given
$\therefore$ $(m\overline{BD})^2 = c^2$	
or $m\overline{BD} = c$	Taking square root of both sides.

Now in	
$\Delta DCB \leftrightarrow \Delta ACB$	
$\overline{CA} \cong \overline{CA}$	Construction
$\overline{BC} \cong \overline{BC}$	Common
$\overline{DB} \cong \overline{AB}$	Each side = c.
$\therefore$ $\Delta DCB \cong \Delta ACB$	S.S.S. ≅ S.S.S.
$\therefore$ $\angle DCB \cong \angle ACB$	(Corresponding angles of congruent triangles)
But $m\angle DCB = 90^0$	Construction
$\therefore$ $\angle ACB = 90^0$	
Hence the ΔACD is a right-angled triangle.	

Corollaries

Let c be the longest of the sides *a, b* and *c* of a triangle.

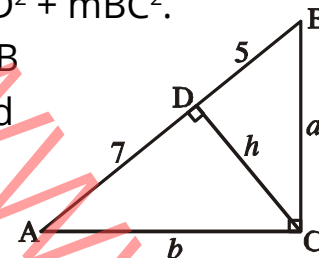
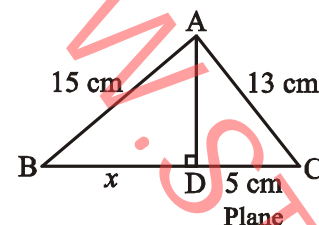
* If $a^2 + b^2 = c^2$, then the triangle is right.
* If $a^2 + b^2 > c^2$, then the triangle is acute.
* If $a^2 + b^2 < c^2$, then the triangle is obtuse.

EXERCISE 15

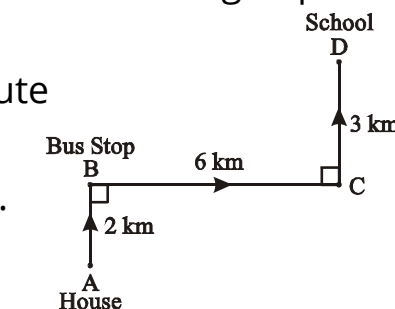
1. Verify that the Δs having the following measures of sides are right - angled.
   (i) a = 5 cm, b = 12 cm, c = 13 cm
   (ii) a = 1.5 cm, b = 2 cm, c = 2.5 cm
   (iii) a = 9 cm, b = 12 cm, c = 15 cm
   (iv) a = 16 cm, b = 30 cm, c = 34 cm
2. Verify that $a^2 + b^2$, $a^2 - b^2$ and 2ab are the measures of the sides of a right angled triangle where a and b are any two real numbers ($a > b$).
3. The three sides of a triangle are of measure 8, x and 17 respectively. For what value of x will it become base of a right angled triangle?
4. In a isosceles Δ, the base m$\overline{BC}$ = 28cm, and m$\overline{AB}$ = m$\overline{AC}$ = 50cm. If m$\overline{AD} \perp$ m$\overline{BC}$, then find

(i) length of $\overline{AD}$ (ii) area of ΔABC

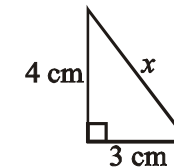
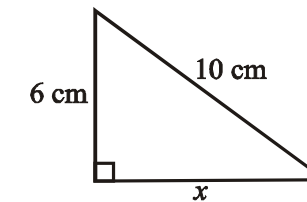
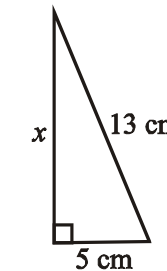
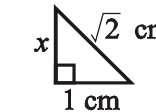
5. In a quadrilateral ABCD, the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular to each other. Prove that $m\overline{AB}^2 + m\overline{CD}^2 = m\overline{AD}^2 + m\overline{BC}^2$.

6. (i) In the ΔABC as shown in the figure, $m\angle ACB = 90^0$ and $\overline{CD} \perp \overline{AE}$ Find the lengths a, h and b if $m\overline{BD}$ = 5 units and $m\overline{AD}$ = 7 units.

(ii) Find the value of x in the shown figure.

7. A plane is at a height of 300 m and is 500 m away from the airport as shown in the figure. How much distance will it travel to land at the airport?

8. A ladder 17 m long rests against a vertical wall. The foot of the ladder is 8 m away from the base of the wall. How high up the wall will the ladder reach?

9. A student travels to his school by the route as shown in the figure. Find $m\overline{AD}$, the direct distance from his house to school.

REVIEW EXERCISE 15

1. Which of the following are true and which are false?
   (i) In a right angled triangle greater angle is of 90^0. ......
   (ii) In a right angled triangle right angle is of 60^0. ......
   (iii) In a right triangle hypotenuse is a side opposite to right angle. ......
   (iv) If a, b, c are sides of right angled triangle with c as longer side, then $c^2 = a^2 + b^2$......
   (v) If 3 cm and 4 cm are two sides of a right angled triangle, then hypotenuse is 5 cm. ......
   (vi) If hypotenuse of an isosceles right triangle is $\sqrt{2}$ cm, then each of other side is of length 2 cm. ......

2. Find the unknown value in each of the following figures.

(i) (ii) (iii) (iv)

SUMMARY

In this unit we learned to state and prove Pythagoras' Theorem and its converse with corollaries.

- In a right angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides.
- If the square of one side of a triangle is equal to the sum of the squares of the other two sides then the triangle is a right angled triangle.

Moreover, these theorems were applied to solve some questions of practical use.

16

THEOREMS RELATED WITH AREA

Animation 16.1: mirandamolina
Source & Credit: The Math Kid

Students Learning Outcomes

After studying this unit, the students will be able to:

- Prove that parallelograms on the same base and lying between the same parallel lines (or of the same altitude) are equal in area.
- Prove that parallelograms on equal bases and having the same altitude are equal in area.
- Prove that triangles on the same base and of the same altitude are equal in area.
- Prove that triangles on equal bases and of the same altitude are equal in area.

Introduction

In this unit we will state and prove some important theorems related with area of parallelograms and triangles along with corollaries. We shall apply them to solve appropriate problems and to prove some useful results.

Some Preliminaries

Area of a Figure

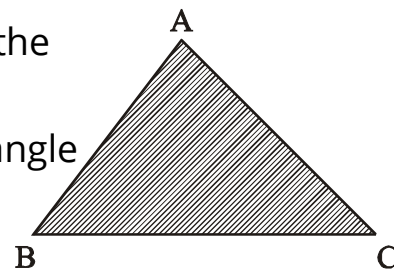
The region enclosed by the bounding lines of a closed figure is called the area of the figure.

The area of a closed region is expressed in square units (say, sq. m or m^2) i.e. a positive real number.

Triangular Region

The interior of a triangle is the part of the plane enclosed by the triangle.

A triangular region is the union of a triangle and its interior i.e., the three line segments forming the triangle and its interior.

By area of a triangle, we mean the area of its triangular region.

Congruent Area Axiom

If $\Delta ABC \cong \Delta PQR$, then area of (region ΔABC) = area of (region ΔPQR)

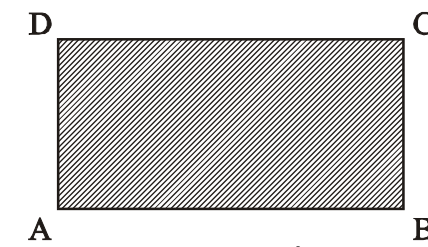
Rectangular Region

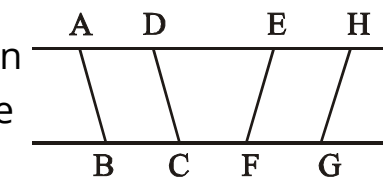
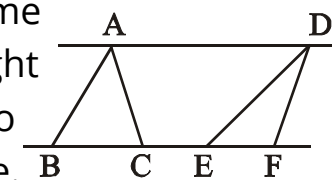
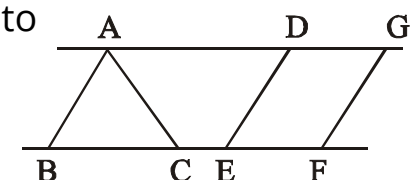
The interior of a rectangle is the part of the plane enclosed by the rectangle.

A rectangular region is the union of a rectangle and its interior.

A rectangular region can be divided into two or more than two triangular regions in many ways.

Recall that if the length and width of a rectangle are ***a*** units and ***b*** units respectively, then the area of the **rectangle** is equal to ***a*** x ***b*** square units.

If a is the side of a square, its area = a^2 square units.

Between the same Parallels

Two parallelograms are said to be between the same parallels, when their bases are in the same straight line and their sides opposite to these bases are also in a straight line; as the parallelograms ABCD, EFGH in the given figure.

Two triangles are said to be between the same parallels, when their bases are in the same straight line and the line joining their vertices is parallel to their bases; as the ∆s ABC, DEF in the given figure.

A triangle and a parallelogram are said to be between the same parallels, when their bases are in the same straight line, and the side of the parallelogram opposite the base, produced if necessary, passes through the vertex of the triangle as are the ∆ABC and the parallelogram DEFG in the given figure.

Definition

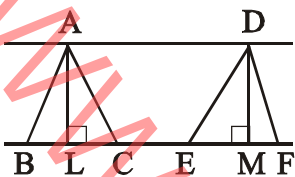
If one side of a parallelogram is taken as its base, the perpendicular distance between that side and the side parallel to it, is called the Altitude or Height of the parallelogram.

Definition

If one side of a triangle is taken as its base, the perpendicular to that side, from the opposite vertex is called the Altitude or Height of the triangle.

Useful Result

Triangles or parallelograms placed between the same or equal parallels will have the same or equal altitudes or heights.

Place the triangles ABC, DEF so that their bases $\overline{BC}$, $\overline{EF}$ are in the same straight line and the vertices on the same side of it, and suppose $\overline{AL}$, $\overline{DM}$ are the equal altitudes. We have to show that $\overline{AD}$ is parallel to BCEF.

Proof

$\overline{AL}$ and $\overline{DM}$ are parallel, for they are both perpendicular to $\overline{BF}$.
Also $m\overline{AL} = m\overline{DM}$. (given)

$\therefore$ $\overline{AD}$ is parallel to $\overline{LM}$.

A similar proof may be given in the case of parallelograms.

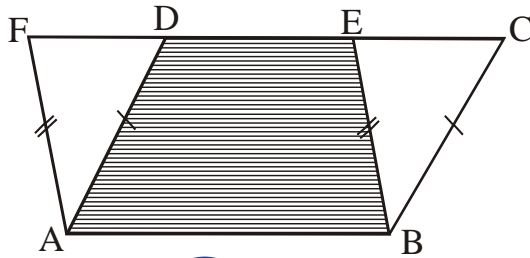
Useful Result

A diagonal of a parallelogram divides it into two congruent triangles (S.S.S.) and hence of equal area.

Theorem 16.1.1

Parallelograms on the same base and between the same parallel lines (or of the same altitude) are equal in area.

Given

Two parallelograms ABCD and ABEF having the same base $\overline{AB}$ and between the same parallel lines $\overline{AB}$ and $\overline{DE}$.

To Prove

area of parallelogram ABCD = area of parallelogram ABEF

Proof

Statements	Reasons
area of (parallelogram ABCD) = area of (quad. ABED) + area of (ΔCBE) ... (1)	[Area addition axiom]
area of (parallelogram ABEF) = area of (quad. ABED) + area of (ΔDAF) ... (2)	[Area addition axiom]
In Δs CBE and DAF	
$m\overline{CB} = m\overline{DA}$	[opposite sides of a parallelogram]
$m\overline{BE} = m\overline{AF}$	[opposite sides of a parallelogram]
$m\angle CBE = m\angle DAF$	[$\because$ $\overline{BC} \parallel \overline{AD}$, $\overline{BE} \parallel \overline{AF}$]
$\therefore$ $\Delta CBE \cong \Delta DAF$	[S.A.S. cong. axiom]
$\therefore$ area of (ΔCBE) = area of (ΔDAF) ...(3)	[cong. area axiom]
Hence area of (parallelogram ABCD) = area of (parallelogram ABEF)	from (1), (2) and (3)

Corollary

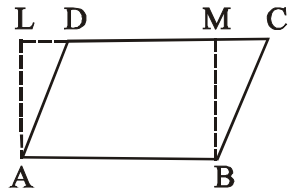
(i) The area of a parallelogram is equal to that of a rectangle on the same base and having the same altitude.

(ii) Hence area of parallelogram = base x altitude

Proof

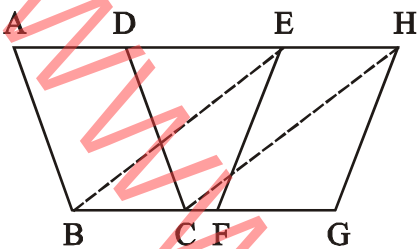
Let ABCD be a parallelogram. $\overline{AL}$ is an altitude corresponding to side $\overline{AB}$.

(i) Since parallelogram ABCD and rectangle ALMB are on the same base $\overline{AB}$ and between the same parallels,
$\therefore$ by above theorem it follows that
area of (parallelogram ABCD) = area of (rect. ALMB)

(ii) But area of (rect. ALMB) = $\overline{AB} \times \overline{AL}$
Hence area of (parallelogram ABCD) = $\overline{AB} \times \overline{AL}$.

Theorem 16.1.2

Parallelograms on equal bases and having the same (or equal) altitude are equal in area.

Given

Parallelograms ABCD, EFGH are on the equal bases $\overline{BC}$, $\overline{FG}$, having equal altitudes.

To Prove

area of (parallelogram ABCD) = area of (parallelogram EFGH)

Construction

Place the parallelograms ABCD and EFGH so that their equal bases $\overline{BC}$, $\overline{FG}$ are in the straight line BCFG. Join $\overline{BE}$ and $\overline{CH}$.

Proof

Statements	Reasons
The given $\|^{gms}$ ABCD and EFGH are between the same parallels	Their altitudes are equal (given)
Hence ADEH is a straight line $\|$ $\overline{BC}$	
∴ $m\overline{BC} = m\overline{FG}$	Given
$= m\overline{EH}$	EFGH is a parallelogram
Now $m\overline{BC} = m\overline{EH}$ and they are $\|$	
∴ $\overline{BE}$ and $\overline{CH}$ are both equal and $\|$	
Hence EBCH is a parallelogram	A quadrilateral with two opposite sides congruent and parallel is a parallelogram
Now Area of $\|^{gm}$ ABCD = Area of $\|^{gm}$ EBCH ....(i)	Being on the same base $\overline{BC}$ and between the same parallels
But Area of $\|^{gm}$ EBCH = Area of $\|^{gm}$ EFGH .....(ii)	Being on the same base $\overline{EH}$ and between the same parallels
Hence area ($\|^{gm}$ ABCD) = area ($\|^{gm}$ EFGH)	From (i) and (ii)

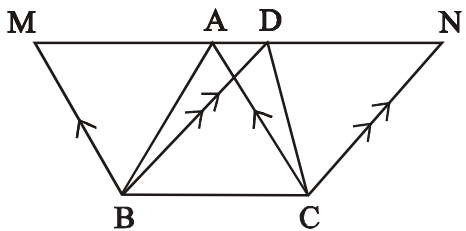
EXERCISE 16.1

1. Show that the line segment joining the mid-points of opposite sides of a parallelogram, divides it into two equal parallelograms.
2. In a parallelogram ABCD, $m\overline{AB}$ = 10 cm. The altitudes corresponding to sides AB and AD are respectively 7 cm and 8 cm. Find $\overline{AD}$.
3. If two parallelograms of equal areas have the same or equal bases, their altitudes are equal.

Theorem 16.1.3

Triangles on the same base and of the same (i.e. equal) altitudes are equal in area.

Given

Δs ABC, DBC on the same base $\overline{BC}$, and having equal altitudes.

To Prove

area of (ΔABC) = area of (ΔDBC)

Construction

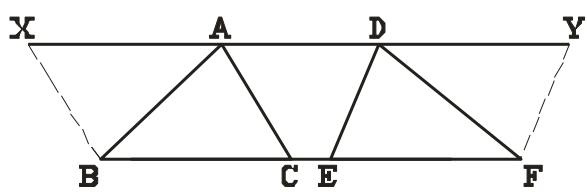
Draw $\overline{BM}$ $\|$ to $\overline{CA}$, $\overline{CN}$ $\|$ to $\overline{BD}$ meeting $\overline{AD}$ produced in M, N.

Proof

Statements	Reasons
Δ ABC and Δ DBC are between the same $\|^s$	Their altitudes are equal
Hence MADN is parallel to $\overline{BC}$	
∴ Area ($\|^{gm}$ BCAM) = Area ($\|^{gm}$ BCND) ....(i)	These $\|^{gm}$ are on the same base $\overline{BC}$ and between the same $\|^s$
But Area of ΔABC = $\frac{1}{2}$(Area of $\|^{gm}$ BCAM) ....(ii)	Each diagonal of a $\|^{gm}$ bisects it into two

and Area of $\Delta DBC = \frac{1}{2}$(Area of ($\|^{gm}$ BCND) ....(iii)	congruent triangles
Hence Area (Δ ABC) = Area (Δ DBC)	From (i), (ii) and (iii)

Theorem 16.1.4

Triangles on equal bases and of equal altitudes are equal in area.

Given

Δs ABC, DEF on equal base $\overline{BC}$, $\overline{EF}$ and having altitudes equal.

To Prove

Area of (Δ ABC) = Area of (Δ DEF)

Construction

Place the Δs ABC and DEF so that their equal bases $\overline{BC}$ and $\overline{EF}$ are in the same straight line BCEF and their vertices on the same side of it. Draw $\overline{BX} \parallel \overline{CA}$ and $\overline{FY} \parallel$ ED meeting $\overline{AD}$ produced in X, Y respectively.

Proof

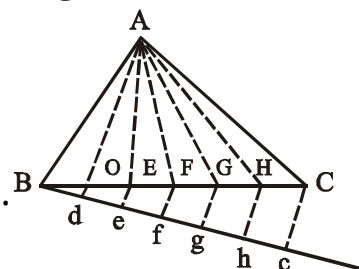
Statements	Reasons
ΔABC, ΔDEF are between the same parallels	Their altitudes are equal (given)
$\therefore$ XADY is $\parallel$ to BCEF	
$\therefore$ area ($\|^{gm}$ BCAX) = area ($\|^{gm}$ EFYD) ......(i)	These $\|^{gm}$ are on equal bases and between the same parallels
But Area of $\Delta ABC = \frac{1}{2}$ Area of ($\|^{gm}$ BCAX) ......(ii)	Diagonal of a $\|^{gm}$ bisects it

and area of $\Delta DFE = \frac{1}{2}$ area of ($\|^{gm}$ EFYD) ......(iii)	
$\therefore$ area (ΔABC) = area (ΔDEF)	From (i), (ii) and (iii)

Corollaries

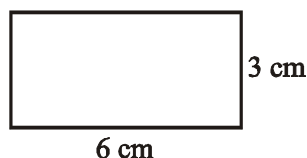
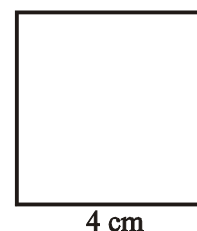
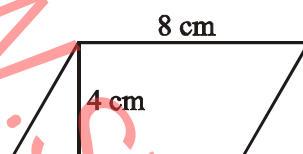
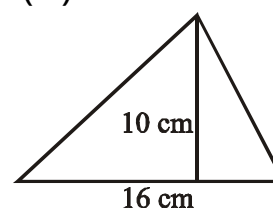
1. Triangles on equal bases and between the same parallels are equal in area.
2. Triangles having a common vertex and equal bases in the same straight line, are equal in area.

EXERCISE 16.2

1. Show that a median of a triangle divides it into two triangles of equal area.
2. Prove that a parallelogram is divided by its diagonals into four triangles of equal area.
3. Divide a triangle into six equal triangular parts.

REVIEW EXERCISE 16

1. Which of the following are true and which are false?
   - (i) Area of a figure means region enclosed by bounding lines of closed figure.
   - (ii) Similar figures have same area.
   - (iii) Congruent figures have same area.
   - (iv) A diagonal of a parallelogram divides it into two non-congruent triangles.
   - (v) Altitude of a triangle means perpendicular from vertex to the opposite side (base).
   - (vi) Area of a parallelogram is equal to the product of base and height.

2. Find the area of the following.

(i)

(ii)

(iii)

(iv)

3. Define the following
   - (i) Area of a figure
   - (ii) Triangular Region
   - (iii) Rectangular Region
   - (iv) Altitude or Height of a triangle

SUMMARY

In this unit we mentioned some necessary preliminaries, stated and proved the following theorems alongwith corollaries, if any.

- Area of a figure means region enclosed by the boundary lines of a closed figure.
- A triangular region means the union of triangle and its interior.
- By area of triangle means the area of its triangular region
- Altitude or height of a triangle means perpendicular distance to base from its opposite vertex.
- Parallelograms on the same base and between the same parallel lines (or of the same altitude) are equal in area.
- Parallelograms on equal bases and having the same (or equal) altitude are equal in area.
- Triangles on the same base and of the same (i.e. equal) altitudes are equal in area.
- Triangles on equal bases and of equal altitudes are equal in area.

CHAPTER 17

PRACTICAL GEOMETRY –TRIANGLES

Animation 17.1: Practical Geometry – Triangles
Source & Credit: eLearn.punjab

Students Learning Outcomes

After studying this unit, the students will be able to:

- Construct a triangle having given: two sides and the included angle, one side and two of the angles, two of its sides and the angle opposite to one of them and two of them angles, two of its sides and the angle opposite to one of them (with all the three possibilities).
- Draw: angle bisectors, altitudes, perpendicular bisectors, medians, of a given triangle and verify their concurrency.
- Construct a triangle equal in area to a given quadrilateral. Construct a rectangle equal in area to a given triangle. Construct a square equal in area to a given rectangle. Construct a triangle of equivalent area on a base of given length.

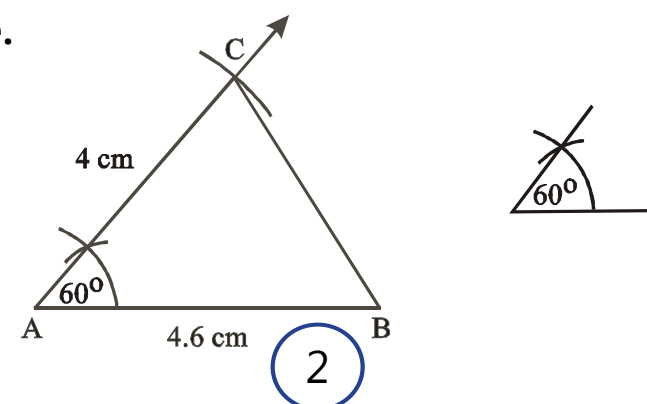
Introduction

In this unit we shall learn to construct different triangles, rectangles, squares etc. The knowledge of these basic constructions is very useful in every day life, especially in the occupations of wood-working, graphic art and metal trade etc. Intermixing of geometrical figures is used to create artistic look. The geometrical constructions are usually made with the help of a pair of compasses, set squares, dividers and a straight edge.

Observe that

If the given line segments are too big or too small , a suitable scale may be taken for constructing the figure.

17.1 Construction of Triangles

(a) To construct a triangle, having given two sides and the included angle.

Given

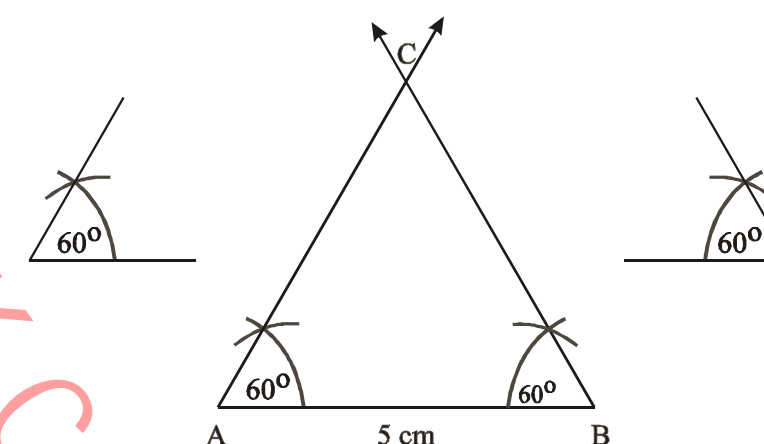
Two sides, say
$m\overline{AB} = 4.6cm$ and $m\overline{AC} = 4cm$ and the included angle, $m\angle A = 60°$.

Required

To construct the $\triangle ABC$ using given information of sides and the included angle = $\angle 60^0$

Construction:

(i) Draw a line segment $m\overline{AB} = 4.6cm$
(ii) At the end A of $\overline{AB}$ make $m\angle BAC = \angle 60^0$
(iii) Cut off $m\overline{AC} = 4cm$ from the terminal side of $\angle 60^0$.
(iv) Join BC
(v) Then ABC is the required $\triangle$.

(b) To construct a triangle, having given one side and two of the angles.

Given

The side $m\overline{AB} = 5cm$, say and two of the angles, say
$m\angle A = 60°$ and $m\angle B = 60°$.

Required

To construct the $\triangle ABC$ using given data.

Construction:

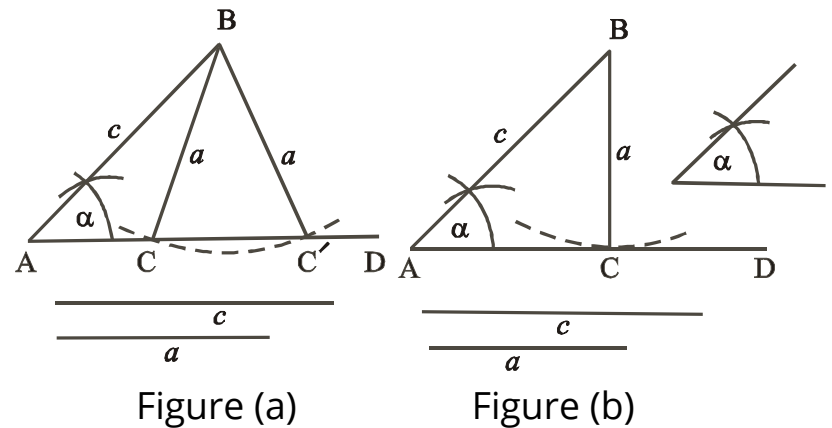
(i) Draw a line segment $m\overline{AB} = 5cm$
(ii) At the end A of $\overline{AB}$ make $m\angle BAC = \angle 60^0$
(iii) At the end point B of $\overline{BA}$ make $m\angle ABC = \angle 60^0$
(iv) The terminal sides of these two angles meet at C.
(v) Then ABC is the required △.

Observe that

When two angles of a triangle are given, the third angle can be found from the fact that the sum of three angles of triangle is 180^0. Thus two angles being known, all the three are known, and we can take any two of these three angles as the base angles with given side as base.

(c) Ambiguous Case

To construct a triangle having given two of its sides and the angle opposite to one of them.

Figure (a) Figure (b)

Given

Two sides a, c and $m\angle A = \alpha$ opposite to one of them, say a.

Required

To construct a triangle having the given parts.

Construction:

(i) Draw a line segment AD of any length.
(ii) At A make $m\angle DAB = m\angle A = \alpha$
(iii) Cut off $\overline{AB} = c$.
(iv) With centre B and radius equal to a, draw an arc.
Three cases arise.

Case I

When the arc with radius a cuts $\overline{AD}$ in two distinct points C and C′ as in Figure (a). Joint $\overline{BC}$ and $\overline{BC}'$.

Then both the triangles ABC and ABC′ have the given parts and are the required triangles.

Case II

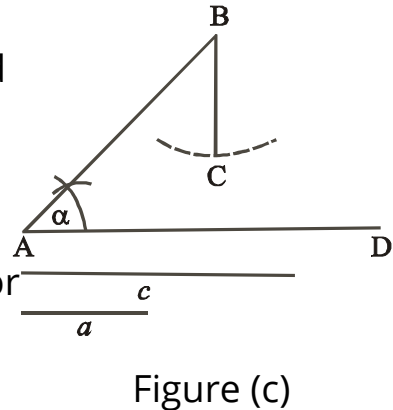
When the arc with radius a only touches $\overline{AD}$ at C, as in Figure (b). Join $\overline{BC}$.

Then △ABC is the required triangle angled at C.

Case III

When the arc with radius a neither cuts nor touches $\overline{AD}$ as in Figure (c).

There will be no triangle in this case.

Figure (c)

Note: Recall that in a △ABC the length of the side opposite to ∠A is denoted by a, opposite to ∠B is denoted by b and opposite to ∠C is denoted by c.

EXERCISE 17.1

1. Construct △ABC in which

(i)	$m\overline{AB} = 3.2cm$,	$m\overline{BC} = 4.2cm$,	$m\overline{CA} = 5.2cm$
(ii)	$m\overline{AB} = 4.2cm$,	$m\overline{BC} = 3.9cm$,	$m\overline{CA} = 3.6cm$
(iii)	$m\overline{AB} = 4.8cm$,	$m\overline{BC} = 3.7cm$,	$m\angle B = 60°$
(iv)	$m\overline{AB} = 3cm$,	$m\overline{AC} = 3.2cm$,	$m\angle A = 45°$

(v) $m\overline{AB} = 4.2$cm, $m\overline{CA} = 3.5$cm, $m\angle C = 75^\circ$
(vi) $m\overline{AB} = 2.5$cm, $m\angle A = 30^\circ$, $m\angle B = 105^\circ$
(vii) $m\overline{AB} = 3.6$cm, $m\angle A = 75^\circ$, $m\angle B = 45^\circ$

2. Construct $\triangle XYZ$ in which
   (i) $m\overline{YZ} = 7.6$cm, $m\overline{XY} = 6.1$cm and $m\angle X = 90^\circ$
   (ii) $m\overline{ZX} = 6.4$cm, $m\overline{YZ} = 2.4$cm and $m\angle X = 90^\circ$
   (iii) $m\overline{XY} = 5.5$cm, $m\overline{ZX} = 4.5$cm and $m\angle Z = 90^\circ$.
3. Construct a right-angled $\triangle$ measure of whose hypotenuse is 5 cm and one side is 3.2 cm. (Hint: Angle in a semi-circle is a right angle).
4. Construct a right-angled isosceles triangle whose hypotenuse is
   (i) 5.2 cm long
   [Hint: A point on the right bisector of a line segment is equidistant form its end points.]
   (ii) 4.8 cm (iii) 6.2 cm (iv) 5.4 cm
5. (Ambiguous Case) Construct a $\triangle ABC$ in which
   (i) $m\overline{AC} = 4.2$cm, $m\overline{AB} = 5.2$cm, $m\angle B = 45^\circ$ (two Δs)
   (ii) $m\overline{AC} = 2.5$cm, $m\overline{AB} = 5.0$cm, $m\angle A = 30^\circ$ (one Δs)
   (iii) $m\overline{BC} = 5$cm, $m\overline{AB} = 3.5$cm, $m\angle B = 60^\circ$

Definitions

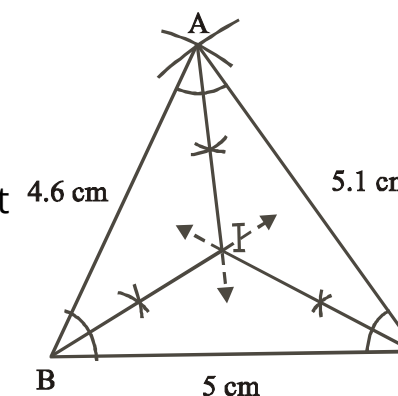
Three or more than three lines are said to be concurrent, if they all pass through the same point. The common point is called the point of concurrency of the lines. The point of concurrency has its own importance in geometry. They are given special names.

(i) The internal bisectors of the angles of a triangle meet at a point called the incentre of the triangle.
(ii) The point of concurrency of the three perpendicular bisectors of the sides of a $\triangle$ is called the circumcentre of the $\triangle$.
(iii) The point of concurrency of the three altitudes of a $\triangle$ is called its orthocentre.
(iv) The point where the three medians of a $\triangle$ meet is called the centroid of the triangle.

17.1.1 Drawing angle bisectors, altitudes etc.

(a) Draw angle bisectors of a given triangle and verify their concurrency.

Example

(i) Construct $\triangle ABC$ having given $m\overline{AB} = 4.6$cm, $m\overline{BC} = 5$cm and $m\overline{CA} = 5.1$cm.
(ii) Draw its angle bisectors and verify that they are concurrent.

Given

The side $m\overline{AB} = 4.6$cm, $m\overline{BC} = 5$cm and $m\overline{CA} = 5.1$cm of a ΔABC.

Required

(i) To construct $\triangle ABC$.
(ii) To draw its angle bisectors and verify their concurrency.

Construction

(i) Take $m\overline{BC} = 5$cm.
(ii) With B as centre and radius $m\overline{BA} = 4.6$cm draw an arc.
(iii) With C as centre and radius $m\overline{CA} = 5.1$cm draw another arc which intersects the first arc at A.
(iv) Join $\overline{BA}$ and $\overline{CA}$ to complete the $\triangle ABC$.
(v) Draw bisectors of $\angle B$ and $\angle C$ meeting each other in the point I.
(vi) Now draw bisector of the third $\angle A$.
(vii) We observe that the third angle bisector also passes through the point I.
(viii) Hence the angle bisectors of the $\triangle ABC$ are concurrent at I, which lies within the $\triangle$.

Note: Recall that the point of concurrency of bisectors of the angles of triangle is called its **incentre.**

(b) Draw altitudes of a given triangle and verify their concurrency.

Example

(i) Construct a triangle ABC in which $m\overline{BC} = 5.9cm$, $m\angle B = 56^0$ and $m\angle C = 44^0$.

(ii) Draw the altitudes of the triangle and verify that they are concurrent.

Given

The side $m\overline{BC} = 5.9cm$ and $m\angle B = 56^0$, $m\angle C = 44^0$.

Required

(i) To Construct △ABC.

(ii) To draw its altitudes and verify their concurrency.

Construction

(i) Take $m\overline{BC} = 5.9cm$.

(ii) Using protractor draw $m\angle CBA = 56^0$ and $m\angle BCA = 44^0$ to complete the △ABC

(iii) From the vertex A drop $\overline{AP} \perp \overline{BC}$.

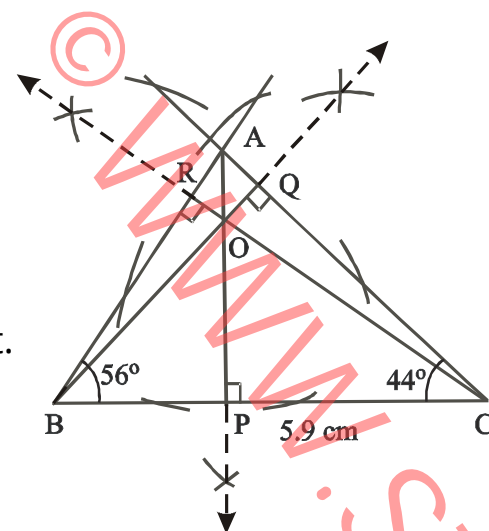
(iv) From the vertex B drop $\overline{BQ} \perp \overline{CA}$. These two altitudes meet in the point O inside the △ABC.

(v) Now from the third vertex C, drop $\overline{CR} \perp \overline{AB}$.

(vi) We observe that this third altitude also passes through the point of intersection O of the first two altitudes.

(vii) Hence the three altitudes of △ABC are concurrent at O.

Note: Recall that the point of concurrency of the three altitudes of a triangle is called its **orthocentre.**

(c) Draw perpendicular bisectors of the sides of a given triangle and verify their concurrency.

Example

(i) Construct a △ABC having given $m\overline{AB} = 4cm$, $m\overline{BC} = 4.8cm$ and $m\overline{AC} = 3.6cm$.

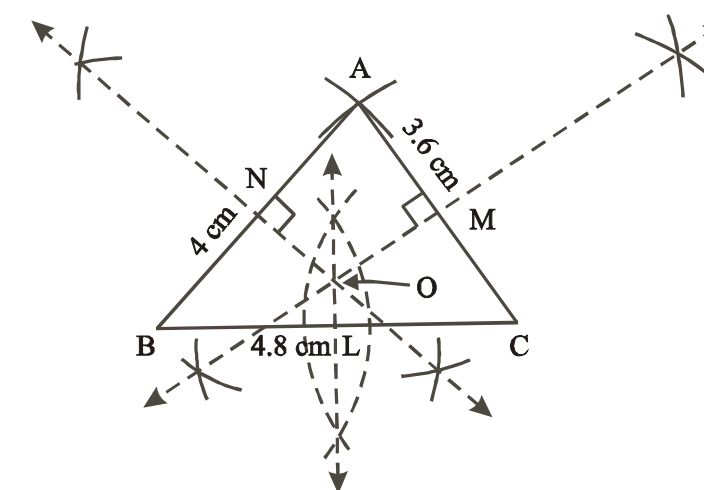
(ii) Draw perpendicular bisectors of its sides and verify that they are concurrent.

Given

Three sides $m\overline{AB} = 4cm$, $m\overline{BC} = 4.8cm$ and $m\overline{AC} = 3.6cm$ of a △ABC.

Required

(i) To Construct △ABC.

(ii) To draw perpendicular bisectors of its sides and to verify that they are concurrent.

Construction

(i) Take $m\overline{BC} = 4.8cm$.

(ii) With B as centre and radius $m\overline{BA} = 4cm$ draw an arc.

(iii) With C as centre and radius $m\overline{CA} = 3.6cm$ draw another arc that intersects the first arc at A.

(iv) Join $\overline{BA}$ and $\overline{CA}$ to complete the △ABC.

(v) Draw perpendicular bisectors of $\overline{BC}$ and $\overline{CA}$ meeting each other at the point O.

(vi) Now draw the perpendicular bisector of third side $\overline{AB}$.

(vii) We observe that it also passes through O, the point of intersection of first two perpendicular bisectors.

(viii) Hence the three perpendicular bisectors of size of $\triangle ABC$ are concurrent at O.

Note: Recall that the point of concurrency of the perpendicular bisectors of the sides of a triangle is called its **circumcentre.**

(d) Draw medians of a given triangle and verify their concurrency

Example

(i) Construct a $\triangle ABC$ in which $m\overline{AB} = 4.8cm$, $m\overline{BC} = 3.5cm$ and $m\overline{AC} = 4cm$.

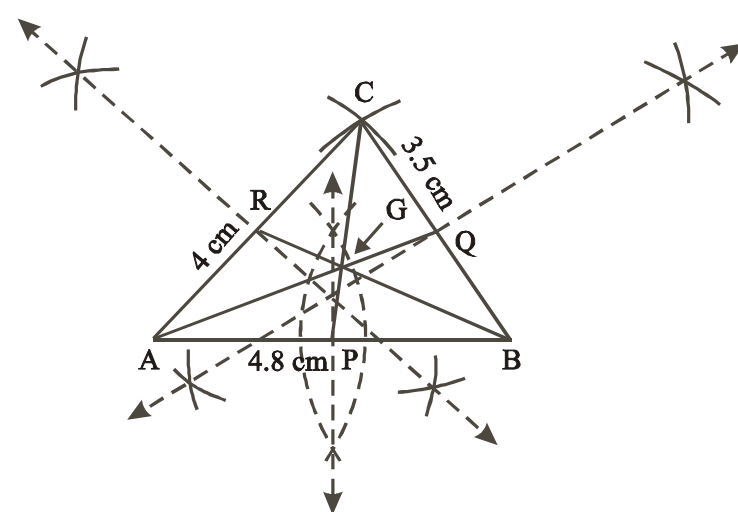
(ii) Draw medians of $\triangle ABC$ and verify that they are concurrent at a point within the triangle. By measurement show that the medians divide each other in the ratio 2 : 1.

Given

Three side $m\overline{AB} = 4.8cm$, $m\overline{BC} = 3.5cm$ and $m\overline{AC} = 4cm$ of a $\triangle ABC$.

Required

(i) To Construct $\triangle ABC$.

(ii) Draw its medians and verify their concurrency.

Construction

(i) Take $m\overline{AB} = 4.8cm$.

(ii) With A as centre and $m\overline{AC} = 4cm$ as radius draw an arc.

(iii) With B as centre and radius $m\overline{BC} = 3.5cm$ draw another arc which intersects the first arc at C.

(iv) Join $\overline{AC}$ and $\overline{BC}$ to get the $\triangle ABC$.

(v) Draw perpendicular bisectors of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ of the $\triangle ABC$ and mark their mid-points P, Q and R respectively.

(vi) Join A to the mid-point Q to get the median $\overline{AQ}$.

(vii) Join B to the mid-point R to get the median $\overline{BR}$.

(viii) The medians $\overline{AQ}$ and $\overline{BR}$ meet in the point G.

(ix) Now draw the third median $\overline{CP}$.

(x) We observe that the third median also passes through the point of intersection G of the first two medians.

(xi) Hence the three medians of the $\triangle ABC$ pass through the same point G. That is, they are concurrent at G. By measuring, $\overline{AG} : \overline{GQ} = 2 : 1$ etc.

Note: Recall that the point of concurrency of the three medians of a triangle is called the **centroid** of the $\triangle ABC$.

EXERCISE 17.2

1. Construct the following $\triangle$'s ABC. Draw the bisectors of their angles and verify their concurrency.
   (i) $m\overline{AB} = 4.5cm$, $m\overline{BC} = 3.1cm$, $m\overline{CA} = 5.2cm$
   (ii) $m\overline{AB} = 4.2cm$, $m\overline{BC} = 6cm$, $m\overline{CA} = 5.2cm$
   (iii) $m\overline{AB} = 3.6cm$, $m\overline{BC} = 4.2cm$, $m\angle B = 75^\circ$.
2. Construct the following $\triangle$'s PQR. Draw their altitudes and show that they are concurrent.
   (i) $m\overline{PQ} = 6cm$, $m\overline{QR} = 4.5cm$, $m\overline{PR} = 5.5cm$
   (ii) $m\overline{PQ} = 4.5cm$, $m\overline{QR} = 3.9cm$, $m\angle R = 45^\circ$
   (iii) $m\overline{RP} = 3.6cm$, $m\angle Q = 30^\circ$, $m\angle P = 105^\circ$.
3. Construct the following triangles ABC. Draw the perpendicular bisectors of their sides and verify their concurrency. Do they meet inside the triangle?
   (i) $m\overline{AB} = 5.3cm$, $m\angle A = 45^\circ$, $m\angle B = 30^\circ$
   (ii) $m\overline{BC} = 2.9cm$, $m\angle A = 30^\circ$, $m\angle B = 60^\circ$
   (iii) $m\overline{AB} = 2.4cm$, $m\overline{AC} = 3.2cm$, $m\angle A = 120^\circ$.

4. Construct the following △s XYZ. Draw their three medians and show that they are concurrent.
   (i) $m\overline{YZ}$ = 4.1cm, $m\angle Y$ = 60°, $m\angle X$ = 75°
   (ii) $m\overline{XY}$ = 4.5cm, $m\overline{YZ}$ = 3.4cm, $m\overline{ZX}$ = 5.6cm
   (iii) $m\overline{ZX}$ = 4.3cm, $m\angle X$ = 75°, $m\angle Y$ = 45°

17.2 Figures with Equal Areas

(i) Construct a triangle equal in area to a given quadrilateral.

Given

A quadrilateral ABCD.

Required

To construct a △ equal in area to quadrilateral ABCD.

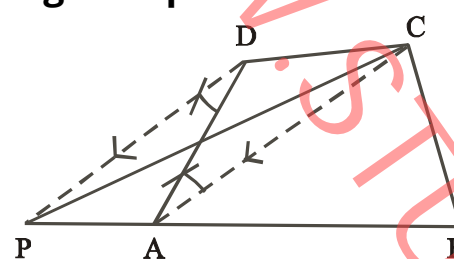
Construction

(i) Join AC.
(ii) Through D draw DP || CA, meeting BA produced at P.
(iii) Join PC.
(iv) Then PBC is the required triangle.

Observe that

△s APC, ADC stand on the same base AC and between the same parallels AC and PD.

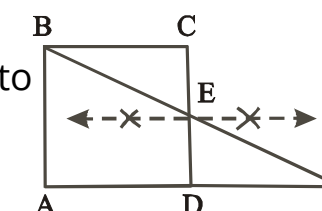
Hence △APC = △ADC

△APC + △ABC = △ADC + △ABC or △PBC = quadrilateral ABCD.

EXERCISE 17.3

1. (i) Construct a quadrilateral ABCD, having $m\overline{AB} = m\overline{AC}$ = 5.3cm, $m\overline{BC} = m\overline{CD}$ = 3.8cm and $m\overline{AD}$ = 2.8cm.
   (ii) On the side BC construct a △ equal in area to the quadrilateral ABCD.
2. Construct a △ equal in area to the quadrilateral PQRS, having

$m\overline{QR}$ = 7cm, $m\overline{RS}$ = 6cm, $m\overline{SP}$ = 2.75cm. $m\angle QRS = 60^0$, and $m\angle RSP = 90^0$.

[Hint: $2.75 = \frac{1}{2} \times 5.5$]

3. Construct a △ equal in area to the quadrilateral ABCD, having $m\overline{AB}$ = 6cm, $m\overline{BC}$ = 4cm, $m\overline{AC}$ = 7.2cm, $m\angle BAD = 105^0$, and $m\overline{BD}$ = 8cm.
4. Construct a right-angled triangle equal in area to a given square.

(ii) Construct a rectangle equal in area to a given triangle.

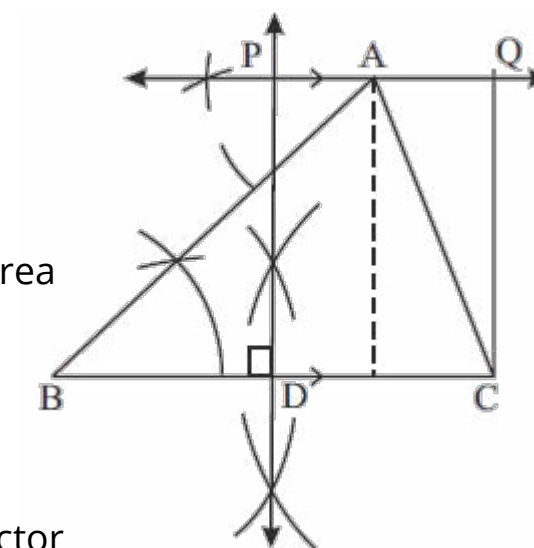
Given

△ABC

Required

To construct a rectangle equal in area to △ABC .

Construction

(i) Take a △ABC.
(ii) Draw $\overleftrightarrow{DP}$, the perpendicular bisector of $\overline{BC}$.
(iii) Through the vertex A of △ABC draw $\overleftrightarrow{PAQ} \parallel \overline{BC}$ intersecting $\overleftrightarrow{PD}$ at P.
(iv) Take $m\overline{PQ} = m\overline{DC}$.
(v) Join Q and C.
(vi) Then CDPQ is the required rectangle.

Example

Construct a parallelogram equal in area to a given triangle having one angle equal to a given angle.

Given

△ABC and ∠α.

Required

To construct a parallelogram equal in area to △ABC and having one angle = ∠α

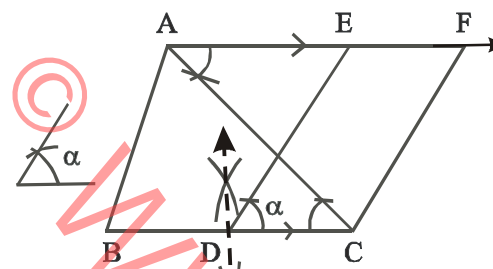
Construction

(i) Bisect $\overline{BC}$ at D.

(ii) Draw $\overline{DE}$ making ∠CDE = ∠α

(iii) Draw $\overrightarrow{AEF}$ || to $\overline{BC}$ cutting $\overline{DE}$ at E.

(iv) Cut off $\overline{EF}$ = $\overline{DC}$. Join C and F.

Then CDEF is the required parallelogram.

EXERCISE 17.4

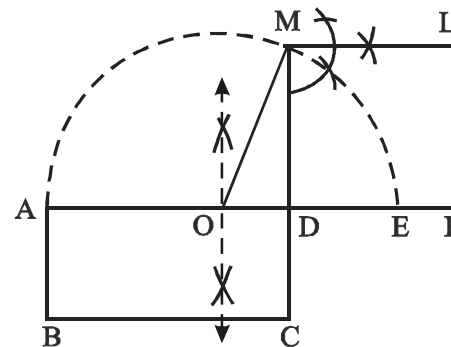
1. Construct a △ with sides 4 cm, 5 cm and 6 cm and construct a rectangle having its area equal to that of the △. Measure its diagonals. Are they equal?
2. Transform an isosceles △ into a rectangle.
3. Construct a △ABC such that m$\overline{AB}$ = 3cm, m$\overline{BC}$ = 3.8cm, m$\overline{AC}$ = 4.8cm. Construct a rectangle equal in area to △ABC, the and measure its sides.

(iii) Construct a square equal in area to a given rectangle.

Given

A rectangle ABCD.

Required

To construct a square equal in area to rectangle ABCD.

Construction

(i) Produced $\overline{AD}$ to E making m$\overline{DE}$ = m$\overline{CD}$.

(ii) Bisect $\overline{AE}$ at O.

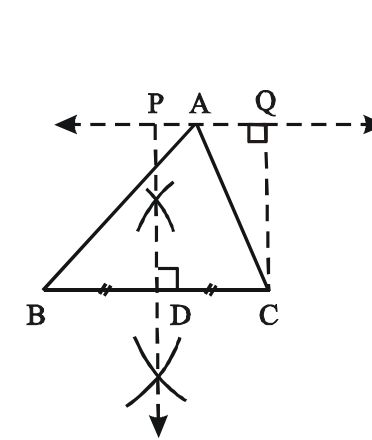
(iii) With centre O and radius $\overline{OA}$ describe a semi - circle.

(iv) Produced $\overline{CD}$ to meet the semi - circle in M.

(v) On $\overline{DM}$ as a side construct a square DFLM.

This shall be the required square.

Example

Construct a square equal in area to a given triangle.

Given

△ABC.

Required

To construct a square equal in area to △ABC.

Construction

(i) Draw $\overleftrightarrow{PAQ}$ || $\overline{BC}$.

(ii) Draw the perpendicular bisector of $\overline{BC}$, bisecting it at D and meeting $\overleftrightarrow{PAQ}$ at P.

(iii) Draw $\overline{CQ}$ ⊥ $\overline{PQ}$ meeting it in Q.

(iv) Take a line EFG and cut off $\overline{EF}$ = $\overline{DP}$ and $\overline{FG}$ = $\overline{DC}$.

(v) Bisect $\overline{EG}$ at O.

(vi) With O as centre and radius = $\overline{OE}$ draw a semi - circle.

(vii) At F draw $\overline{FM}$ ⊥ $\overline{EG}$ meeting the semi - circle at M.

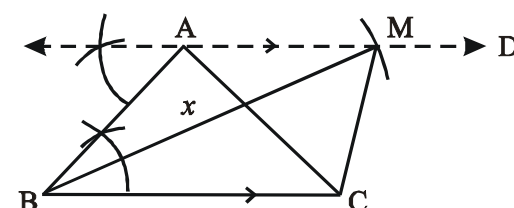
(viii) With $\overline{MF}$ as a side, complete the required square FMNR.

(iv) Construct a triangle of equivalent area on a base of given length.

Given

$\triangle ABC$

Required

To construct a triangle with base x and having area equivalent to area $\triangle ABC$.

Construction

(i) Construct the given $\triangle ABC$.
(ii) Draw $\overleftrightarrow{AD} \parallel \overline{BC}$.
(iii) With B as centre and radius = x, draw an arc cutting $\overleftrightarrow{AD}$ in M.
(iv) Join $\overline{BM}$ and $\overline{CM}$.
(v) Then BCM is the required triangle with base $\overline{BM} = x$ and area equivalent to $\triangle ABC$.

EXERCISE 17.5

1. Construct a rectangle whose adjacent sides are 2.5 cm and 5 cm respectively. Construct a square having area equal to the given rectangle.
2. Construct a square equal in area to a rectangle whose adjacent sides are 4.5 cm and 2.2 cm respectively. Measure the sides of the square and find its area and compare with the area of the rectangle.
3. In Q.2 above verify by measurement that the perimeter of the square is less than that of the rectangle.
4. Construct a square equal in area to the sum of two squares having sides 3 cm and 4 cm respectively.
5. Construct a $\triangle$ having base 3.5 cm and other two sides equal to 3.4 cm and 3.8 cm respectively. Transform it into of a square equal square area.
6. Construct a $\triangle$ having base 5 cm and other sides equal to 5 cm and 6 cm. Construct a square equal in area to given $\triangle$.

REVIEW EXERCISE 17

1. Fill in the following blanks to make the statement true:
(i) The side of a right angled triangle opposite to 90° is called ......
(ii) The line segment joining a vertex of a triangle to the mid-point of its opposite side is called a ......
(iii) A line drawn from a vertex of a triangle which is ...... to its opposite side is called an altitude of the triangle.
(iv) The bisectors of the three angles of a triangle are ......
(v) The point of concurrency of the right bisectors of the three sides of the triangle is ...... from its vertices.
(vi) Two or more triangles are said to be similar if they are equiangular and measures of their corresponding sides are ....
(vii) The altitudes of a right triangle are concurrent at the ...... of the right angle.

2. Multiple Choice Questions. Choose the correct answer.

3. Define the following
(i) Incentre (ii) Circumcentre
(iii) Ortho centre (iv) Centroid
(v) Point of concurrency

SUMMARY

In this unit we learnt the construction of following figures and relevant concepts:

- To construct a triangle, having given two sides and the included angle.

- To construct a triangle, having given one side and two of the angles.
- To construct a triangle having given two of its sides and the angle opposite to one of them.
- Draw angle bisectors of a given triangle and verify their concurrency.
- Draw altitudes of a given triangle and verify their concurrency.
- Draw perpendicular bisectors of the sides of a given triangle and verify their concurrency.
- Draw medians of a given triangle and verify their concurrency.
- Construct a triangle equal in area to a given quadrilateral.
- Construct a rectangle equal in area to a given triangle.
- Construct a square equal in area to a given rectangle.
- Construct a triangle of equivalent area on a base of given length.
- Three or more than three lines are said to be concurrent if these pass through the same point and that point is called the point of concurrency.
- The point where the internal bisectors of the angles of a triangle meet is called incentre of a triangle.
- Circumscentre of a triangle means the point of concurrency of the three perpendiculars bisectors of the sides of a triangle.
- Median of a triangle means a line segment joining a vertex of a triangle to the midpoint of the opposite side.
- Orthocentre of a triangle means the point of concurrency of three altitudes of a triangle.